Advanced Calculus I

A Definitions and examples

1. Continuity
   (a) State the definition of "$f\colon (0,1) \to \mathbf{R}$ is continuous".
   (b) Give a concrete example of a continuous function.
   (c) Give a concrete example of a function that is not continuous.
2. Differentiability
   (a) State the definition of "$f\colon (0,1) \to \mathbf{R}$ is differentiable".
   (b) Give a concrete example of a function that is differentiable.
   (c) Give a concrete example of a function that is not differentiable.
3. Integrability
   (a) State the definition of "$f\colon [0,1] \to \mathbf{R}$ is Riemann integrable".
   (b) Give a concrete example of a function that is Riemann integrable.
   (c) Give a concrete example of a function that is not Riemann integrable.

B Theorems and proofs

Here are some of the important theorems from the course:

- Bolzano–Weierstrass theorem
- Intermediate-value theorem
- Mean-value theorem
- Taylor's formula
- l'Hôpital's rule
- Fundamental theorem of calculus

4. Give careful statements of *three* of the indicated theorems.
   (For a theorem that has several versions, state any one version.)
5. Prove *one* of the indicated theorems.
   (For a theorem that has several versions, prove any one version.)

Advanced Calculus I

C Problems

Solve *two* of the following four problems.

6. Prove that $\sum_{k=1}^{n} k^3 = \dfrac{n^2(n+1)^2}{4}$ for every natural number n.

7. Prove that $\left\{n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence.

8. Define $f\colon (0,\infty) \to (0,\infty)$ by setting $f(x)$ equal to xe^x for each positive real number x. Prove that f has an inverse function, and evaluate the derivative $(f^{-1})'(e)$.

9. Let a_n equal $\displaystyle\int_1^n \frac{\sin(x)}{\sqrt{x}}\,dx$ for each natural number n. Prove that $\lim_{n\to\infty} a_n$ exists.