Advanced Calculus I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "determine" or "true/false" or "give an example" requires a supporting explanation. Please explain your reasoning in complete sentences.

1. Determine the largest natural number n such that $|n^2 - 2| < 18$.

   Solution. By the definition of absolute value, the given inequality is equivalent to the following double inequality:

   $$-18 < n^2 - 2 < 18 \qquad \text{or} \qquad -16 < n^2 < 20.$$

   Since the square of a natural number is never negative, an equivalent statement is that $n^2 < 20$. The square function is increasing on the natural numbers (by a homework exercise), so it is evident that natural numbers 5 and larger fail to satisfy the inequality, while the natural numbers 1, 2, 3, and 4 do satisfy the inequality. Thus the largest natural number that satisfies the inequality is 4.

2. Let E denote the set of real numbers having decimal expansions

   $$0.a\,b\,c\,d\,\ldots$$

   that do *not* contain the digit 5. Determine the supremum of the set E.

   Solution. The supremum of E is equal to 1. To see why, first observe that 1 is an upper bound for the set of *all* decimal expansions $0.a\,b\,c\,d\,\ldots$ and so too for the subset E (by the monotonicity property of suprema). To show that 1 is the *least* upper bound, it suffices to find an increasing sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E tending to 1. Take x_n equal to $1 - 10^{-n}$. Evidently this sequence is increasing and has limit 1. Moreover, each x_n is an element of E because the decimal expansion of $1 - 10^{-n}$ does not contain the digit 5. (Notice that a real number like 0.99 can be written either as $0.99000\ldots$ or as $0.98999\ldots$, but this ambiguity in the decimal representation has no impact on the preceding argument.)

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3. Suppose that a sequence $\{x_n\}_{n\in\mathbf{N}}$ of real numbers is defined recursively as follows:
$$x_1 = 1, \qquad \text{and}$$
$$x_{n+1} = \sqrt{6 + x_n} \qquad \text{when } n \in \mathbf{N}. \tag{$\dagger$}$$
(The square-root symbol indicates the *positive* square root.)

   Prove that $x_n < 3$ for every natural number n.

   Solution. The proof is by induction. Since $x_1 = 1 < 3$, the basis step holds. Suppose, then, that $x_n < 3$ for a certain natural number n; what needs to be shown is that $x_{n+1} < 3$. Adding 6 to both sides of the induction hypothesis shows that $6 + x_n < 9$. By a homework exercise, the square-root function is increasing on positive real numbers, so $x_{n+1} = \sqrt{6 + x_n} < \sqrt{9} = 3$. Thus $x_{n+1} < 3$, as required. This conclusion completes the induction proof.

4. Give an example of a bijective function $f\colon \mathbf{R} \to (0, \infty)$.
[Recall that "bijective" means both one-to-one and onto.]

   Solution. Perhaps the simplest example is the exponential function: $f(x) = e^x$. You know from your first calculus course that this function has an inverse function (the natural logarithm), hence is bijective.

   Many other examples can be built using tools from your first calculus course. The following piecewise-defined function is one such example:
$$g(x) = \begin{cases} \dfrac{1}{1 + |x|}, & \text{if } x < 0, \\ 1 + x, & \text{if } x \geq 0. \end{cases}$$

   Here is an example that uses only notions from within this course. Express the domain $\mathbf{R}$ as the union $\bigcup_{n\in\mathbf{Z}}(n-1, n]$ of pairwise disjoint, congruent intervals and the codomain $(0, \infty)$ similarly as the union $\bigcup_{n\in\mathbf{N}}(n-1, n]$. The set of integers is countable, so there exists a bijective function $h\colon \mathbf{Z} \to \mathbf{N}$. For each integer n, map the interval $(n-1, n]$ bijectively to the interval $(h(n) - 1, h(n)]$ by the translation that sends each x in $(n-1, n]$ to $h(n) - n + x$. This piecewise-defined function maps $\mathbf{R}$ bijectively onto $(0, \infty)$.

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5. True or false: If E is an uncountable subset of $\mathbf{R}$, then the complement $\mathbf{R} \setminus E$ is a countable set.

   Solution. The statement is false. For instance, if $E = (0, \infty)$, then E is uncountable; the complementary set $\mathbf{R} \setminus E$ is $(-\infty, 0]$, which too is uncountable.

   Here are some more details. Every infinite subset of a countable set is countable, so every superset of an uncountable set is uncountable. We proved in class by Cantor's diagonal argument that the interval $(0, 1)$ is uncountable. Therefore the superset $(0, \infty)$ is uncountable. The reflection $x \mapsto -x$ is a bijection between $(0, \infty)$ and $(-\infty, 0)$, so the latter set is uncountable. Therefore the superset $(-\infty, 0]$ is uncountable too.

6. (a) State the definition of what "$\lim\limits_{n\to\infty} x_n = L$" means.

   Solution. By definition, "$\lim\limits_{n\to\infty} x_n = L$" means that for every positive real number ε, there exists a natural number N such that $|x_n - L| < \varepsilon$ whenever $n \ge N$.

   (b) Use the definition to prove that $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$.

   Solution. Let ε be an arbitrary positive real number; there is no loss of generality in assuming that $\varepsilon < 1$. Set N equal to the natural number $\lfloor 1/\varepsilon \rfloor$. Then $N + 1 > 1/\varepsilon$. If $n \ge N$, then $n + 1 > 1/\varepsilon$, so

   $$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \varepsilon.$$

   Since the positive number ε is arbitrary, it follows from the definition of limit that $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$.

7. Prove that the sequence (†) defined in problem 3 converges.
   [For the purposes of this problem, you may assume that the conclusion of problem 3 is valid. The value of $\lim_{n\to\infty} x_n$ actually is equal to 3, but you are not required to prove that.]

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Solution. According to problem 3, the sequence is bounded above. If it can be shown additionally that the sequence is monotonically increasing, then a theorem that we proved in class implies that the sequence converges (moreover, the limit is the supremum of the sequence). Thus what needs to be checked is that $x_n < x_{n+1}$ for every natural number n.

This monotonicity can be verified by induction. Since $x_1 = 1$, and $x_2 = \sqrt{7} > 1$, the basis step evidently holds. Suppose, then, that $x_n < x_{n+1}$ for a certain natural number n; what needs to be shown is that $x_{n+1} < x_{n+2}$. Adding 6 to both sides of the induction hypothesis shows that $6 + x_n < 6 + x_{n+1}$. The monotonicity of the square-root function on positive real numbers implies that $\sqrt{6 + x_n} < \sqrt{6 + x_{n+1}}$, that is, $x_{n+1} < x_{n+2}$. This conclusion completes the induction proof that the sequence is strictly increasing.

Instead of using induction, one can give a direct proof of the monotonicity as follows. Since $x_n < 3$ (by problem 3), it follows by multiplying both sides by 2 that $2x_n < 6$. Adding x_n to both sides shows that $3x_n < 6 + x_n$. Starting again from the property that $x_n < 3$ and multiplying both sides by the positive number x_n shows that $x_n^2 < 3x_n$. Combining the preceding two deductions by using transitivity of the order relation shows that $x_n^2 < 6 + x_n$. The square-root function is strictly increasing on positive real numbers (by a homework exercise), so $x_n < \sqrt{6 + x_n} = x_{n+1}$. Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is indeed strictly increasing.