Advanced Calculus I

Instructions Please write your solutions on your own paper.

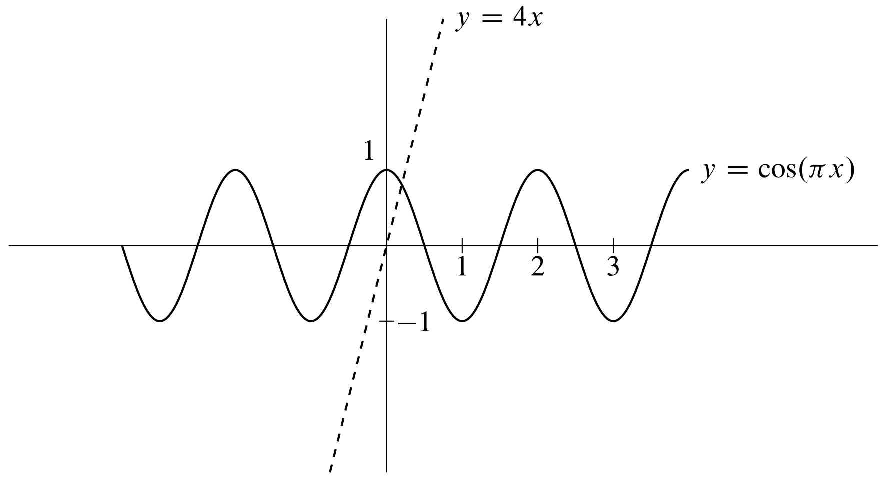
These problems should be treated as essay questions. A problem that says "give an example" or "determine" requires a supporting explanation. In all problems, you should explain your reasoning in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B.

Students in Section 200 (the honors section) should answer questions 1–3 in Part A and questions 7–9 in Part C.

Part A, for both Section 200 and Section 501

1. The diagram below provides convincing evidence that there is exactly one solution in the real numbers to the equation $\cos(\pi x) = 4x$. But a picture is not a proof.



   Your task is to supply a proof, as follows.

   a) Apply the intermediate-value theorem to prove that there is at least one real number x between 0 and 1 such that $\cos(\pi x) - 4x = 0$.

   b) Apply Rolle's theorem (or the mean-value theorem) to prove that there cannot be two distinct real numbers for which $\cos(\pi x) - 4x = 0$.

2. Suppose

$$f(x) = \begin{cases} \log(\cos(\sin(x))), & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

   Is the function f continuous at the point where $x = 0$? Explain why or why not.
   (You may assume that the logarithm function and the trigonometric functions are continuous on their natural domains.)

Advanced Calculus I

3. Suppose a is a positive real number, and

$$f_a(x) = \begin{cases} \dfrac{\sin(x) - x\cos(x)}{|x|^a}, & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

   Show that f_a is differentiable at 0 when $a \leq 2$.

Part B, for Section 501 only

4. The following table has three missing entries: $f'(1)$, $g'(1)$, and $g'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	1	1		
2	2	1	5	

   Determine the missing values if

$$(f \circ g)'(1) = 0,$$
$$(f \circ g)'(2) = 36,$$
$$(g \circ f)'(2) = 45.$$

5. Give an example of a function $f\colon (0, 1) \to \mathbb{R}$ that is increasing, convex, and not uniformly continuous.

6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 0} f(x^2)$ exists but $\lim_{x\to 0} f(x)$ does not exist.

Part C, for Section 200 only

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ for which there are infinitely many real numbers a with the property that $\liminf_{x\to a-} f(x) > \limsup_{x\to a+} f(x)$ (in other words, the limit inferior on the left-hand side exceeds the limit superior on the right-hand side).

8. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ for which the four Dini derivates at the origin all have different values from each other.

9. Show that if $f : \mathbb{R} \to \mathbb{R}$ is convex, and f' (the first derivative) exists everywhere, then f' is necessarily continuous.
   Hint: Can a derivative ever have a jump discontinuity?