Examination 2

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B.

Students in Section 200 should answer questions 1–3 in Part A and questions 7–9 in Part C.

Part A, for both Section 501 and Section 200

1. Show that the function e^{-x} has exactly one fixed point. In other words, there is exactly one real number x with the property that $e^{-x} = x$.

   Solution. Let $f(x)$ denote $e^{-x} - x$; the goal is to show that there is exactly one value of x for which $f(x) = 0$. Since $f(0) = e^0 - 0 = 1 > 0$, and $f(1) = e^{-1} - 1 = (1-e)/e < 0$ (because $e > 1$), the intermediate-value theorem implies that there exists a value of x between 0 and 1 for which $f(x) = 0$. Moreover, the derivative $f'(x)$ equals $-e^{-x} - 1$, which is always negative, so the function f is strictly decreasing. Therefore there cannot be more than point at which $f(x)$ equals 0.

2. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is a one-to-one correspondence (in other words, has an inverse function) but is not differentiable.

   Solution. One example is this piecewise-defined function:

$$\begin{cases} x, & \text{if } x < 0, \\ 2x, & \text{if } x \ge 0. \end{cases}$$

This function is strictly increasing, hence one-to-one. The function is continuous because the two clauses of the definition match when $x = 0$. The function is not differentiable at the origin, for the left-hand derivative equals 1, and the right-hand derivative equals 2.

Another example is the function $x^{1/3}$. This function is strictly increasing but has a vertical tangent line at the origin. Therefore the graph does not have a well-defined slope at the point where $x = 0$.

3. Suppose $f(x) = \tan(\sin(\cos(x)))$. Is there a value of x for which $f'(x) = 0$? Explain why or why not.

Solution. Yes, there is such a value of x.

Method 1. The cosine function has the same value at 0 as at 2π, so $f(0) = f(2\pi)$. By Rolle's theorem, there is a value of x between 0 and 2π for which $f'(x) = 0$.

Method 2. By the chain rule,

$$f'(x) = \sec^2(\sin(\cos(x)))\cos(\cos(x))(-\sin(x)).$$

Since $\sin(0) = 0$, also $f'(0) = 0$.

Part B, for Section 501 only

4. Give an example of a strictly increasing, bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$.

Solution. One example is $\arctan(x)$, which has positive derivative (hence is strictly increasing) and has values bounded between $-\pi/2$ and $\pi/2$.

Another example is $\dfrac{x}{1+|x|}$. When $x > 0$, this expression equals $\dfrac{x}{1+x}$, so the values are bounded between 0 and 1, and the derivative is positive. The original function is odd (antisymmetric), so when $x < 0$, the values are bounded between -1 and 0, and the derivative is again positive.

5. Does the limit

$$\lim_{x\to 0} \frac{\sin(x)\cos(x^2)}{|x|}$$

exist? Explain why or why not.

Solution. The limit does not exist, for the two one-sided limits do not match.

Indeed, since $\cos(x^2)$ is a continuous function that equals 1 when $x = 0$, existence of the indicated limit is equivalent to existence of

$$\lim_{x\to 0} \frac{\sin(x)}{|x|}.$$

For the right-hand limit, observe that

$$\lim_{x\to 0+} \frac{\sin(x)}{|x|} = \lim_{x\to 0+} \frac{\sin(x)}{x} = 1$$

(a limit that we computed in class by applying l'Hôpital's rule). When x is negative, $|x| = -x$, so the left-hand limit admits a similar calculation:

$$\lim_{x\to 0-} \frac{\sin(x)}{|x|} = \lim_{x\to 0-} \frac{\sin(x)}{-x} = -1.$$

Since the one-sided limits are not the same, the two-sided limit does not exist.

Examination 2

6. The following table has two missing entries: $f'(1)$ and $f'(2)$.

x	$f(x)$	$f'(x)$
1	2	
2	3	
3	4	5

Determine the two missing values if

$$(f \circ f)'(1) = 6, \quad \text{and}$$
$$(f^{-1})'(3) = 7.$$

Remember that the notation f^{-1} means the inverse function, not the reciprocal function.

Solution. By the chain rule,

$$6 = (f \circ f)'(1) = f'(f(1))f'(1) = f'(2)f'(1).$$

The rule for inverse functions implies that

$$7 = (f^{-1})'(3) = (f^{-1})'(f(2)) = \frac{1}{f'(2)}.$$

The second equation shows that $f'(2) = 1/7$, and the first equation then implies that $f'(1) = 42$.

Part C, for Section 200 only

7. Give an example of a differentiable, convex function $f : \mathbb{R} \to \mathbb{R}$ for which the second derivative $f''(0)$ does not exist.

Solution. One example is $|x|^{3/2}$. When $x \neq 0$, the derivative equals $\frac{3}{2}|x|^{1/2}\operatorname{sign}(x)$. At the origin, the derivative equals

$$\lim_{x\to 0} \frac{|x|^{3/2} - 0}{x - 0} = \lim_{x\to 0} |x|^{1/2}\operatorname{sign}(x) = 0.$$

Accordingly, the derivative exists everywhere and is strictly increasing, so the original function is convex. The limit defining $f''(0)$ is

$$\lim_{x\to 0} \frac{\frac{3}{2}|x|^{1/2}\operatorname{sign}(x) - 0}{x - 0} = \lim_{x\to 0} \frac{3/2}{|x|^{1/2}},$$

and this limit evidently is $+\infty$.

Another example is the following piecewise-defined function:

$$\begin{cases} 0, & \text{when } x < 0, \\ x^2, & \text{when } x \geq 0. \end{cases}$$

The first derivative of this function equals

$$\begin{cases} 0, & \text{when } x < 0, \\ 2x, & \text{when } x \geq 0. \end{cases}$$

Therefore the right-hand second derivative equals 2, but the left-hand second derivative equals 0. The function is convex, but the second derivative is not defined when $x = 0$.

8. Let $\lceil x \rceil$ denote the ceiling function: namely, the smallest integer greater than or equal to x. Discuss

$$\limsup_{x\to\infty} \frac{\lceil x \rceil \, \cos(x)}{\sqrt{1 + x^2}}.$$

Solution. The quantity $\lceil x \rceil$ is between x and $x + 1$, and so is the quantity $\sqrt{1 + x^2}$ when $x > 0$. Therefore

$$\frac{x}{x+1} \le \frac{\lceil x \rceil}{\sqrt{1+x^2}} \le \frac{x+1}{x},$$

and the squeeze theorem implies that

$$\lim_{x\to\infty} \frac{\lceil x \rceil}{\sqrt{1+x^2}} = 1.$$

Accordingly, if $\{x_n\}$ is a sequence of real numbers tending to ∞ for which $\lim_{n\to\infty} \cos(x_n)$ exists, then

$$\lim_{n\to\infty} \frac{\lceil x_n \rceil \, \cos(x_n)}{\sqrt{1+x_n^2}} = \lim_{n\to\infty} \cos(x_n).$$

The largest possible limit of a convergent sequence of the form $\{\cos(x_n)\}$ is 1 (which is attained when $x_n = 2\pi n$, for example). Therefore the answer to the original problem is 1.

The problem does not ask about the lim inf, but similar reasoning shows that the lim inf equals -1.

9. Show by a counterexample that the analogue of the chain rule fails for right-hand derivatives: namely, there exist continuous functions f and g that do have right-hand derivatives, yet

$$(f \circ g)'_+(0) \ne f'_+(g(0))g'_+(0).$$

Solution. Here is one example: $f(x) = |x|$ and $g(x) = -x$. The composite function $f \circ g$ is the absolute-value function again, so $(f \circ g)'_+(0) = +1$. On the other hand,

$$f'_+(g(0))g'_+(0) = f'_+(0)g'_+(0) = (+1)(-1) = -1.$$