Advanced Calculus I

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "give an example" or "determine" requires a supporting explanation. In all problems, you should explain your reasoning in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B.

Students in Section 200 (the honors section) should answer questions 1–3 in Part A and questions 7–9 in Part C.

Part A, for both Section 200 and Section 501

1. Prove by induction that

$$(1+x)^n \ge 1 + nx \qquad \text{when } -1 < x < 0 \text{ and } n \in \mathbb{N}.$$

2. (a) State the definition of what "$\lim_{n\to\infty} x_n = L$" means.

   (b) Use the definition to prove that $\lim_{n\to\infty} \dfrac{n}{n^2+1} = 0$.

3. Give a concrete example of a non-empty set of real numbers that has no accumulation point.

Part B, for Section 501 only

4. Determine the supremum of the set $\{\, x \in \mathbb{R} : |x^2 - 3| < 6 \,\}$.

5. Suppose that a sequence $\{x_n\}$ is defined recursively as follows:

$$x_1 = 1, \qquad \text{and} \quad x_{n+1} = \frac{1}{1+x_n} \quad \text{when } n \ge 1.$$

   Does this sequence have a convergent subsequence? Explain why or why not.

6. Give a concrete example of a bounded, non-empty set of real numbers that has no interior points and is not compact.

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Part C, for Section 200 only

7. (a) When E is a countable set of real numbers, is the complementary set $\mathbb{R} \setminus E$ always uncountable?

   (b) When E is an uncountable set of real numbers, is the complementary set $\mathbb{R} \setminus E$ always countable?

   Explain why or why not.

8. Does the limit superior respect absolute value? In other words, is $\left|\limsup_{n\to\infty} x_n\right|$ always equal to $\limsup_{n\to\infty} |x_n|$ when $\{x_n\}$ is a sequence of real numbers? Prove or give a counterexample, whichever is appropriate.

9. Consider the following property that a set E of real numbers might or might not have:

   > Whenever a family of *closed* sets covers the set E, some finite subcollection of those closed sets covers E.

   (The Heine–Borel property is the corresponding statement for coverings by *open* sets.) Is this new covering property equivalent to compactness? Explain.