Examination 1

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B.

Students in Section 200 should answer questions 1–3 in Part A and questions 7–9 in Part C.

Part A, for both Section 501 and Section 200

1. Give an example of a set E such that the supremum of E equals 5, but E does not have a maximum.

2. a) State the definition of what "$\lim_{n\to\infty} x_n = \infty$" means.

   b) Use the definition to prove that $\lim_{n\to\infty} \dfrac{n-5}{2} = \infty$.

3. If E is the set of positive irrational numbers less than $5/2$, then what is the set of interior points of E? Explain.

Part B, for Section 501 only

4. The following items are named after famous Greek, Bohemian, German, and French mathematicians:

   - the Archimedean property,
   - the Bolzano–Weierstrass theorem,
   - the Cauchy criterion for convergence.

   Give a precise statement of *one* of these three items.

Examination 1

5. Consider the sequence whose nth term is

$$\frac{(-1)^{5n} + \cos(5n)}{n+5}.$$

Prove that this sequence converges.

6. True or false: If E is a finite set (that is, a set having only a finite number of elements), then E is necessarily compact.
If the statement is true, then give a proof; if false, then give a counterexample.

Part C, for Section 200 only

7. Here are three famous results encountered in the course so far:
   - the well-ordering property of the natural numbers,
   - Cantor's theorem about uncountability,
   - the Heine–Borel characterization of compact sets.

   Give a precise statement of *one* of these three items.

8. Consider the following recursively defined sequence:

$$x_1 = 2, \qquad \text{and} \qquad x_{n+1} = \sqrt{2 + x_n^2} \quad \text{when } n \geq 1.$$

Prove that this sequence has no convergent subsequence.

9. True or false: A set E (a subset of $\mathbb{R}$) is bounded if and only if the closure of E is compact.
If the statement is true, then give a proof; if false, then give a counterexample.