Examination 2

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Give an example of a function, defined on the closed interval $[0, 1]$, that attains a maximum but does not have the intermediate-value property (Darboux property).

2. Suppose f is a twice-differentiable function. Determine
$$\lim_{x \to 1} \frac{f(f(x)) - f(x)}{f(x) - 1}$$
given the information in the following table.

x	$f(x)$	$f'(x)$	$f''(x)$
0	2	1	0
1	1	0	2
2	0	2	1

3. State one of the following items (extra credit for correctly stating both):
   a) Cauchy's version of the mean-value theorem; or
   b) the definition of what it means for a function f to be uniformly continuous on an interval.

4. A function $f(x)$ taking only positive values is called *logarithmically convex* when $\log f(x)$ is a convex function.

   (Here log denotes the natural logarithm function, but the definition is actually independent of the base of the logarithm as long as the base is greater than 1.)

   Show that if $f(x)$ is logarithmically convex, then $f(x)$ must be convex. You may assume that $f(x)$ is twice differentiable.

5. Prove one of the following statements (extra credit for proving both):
   a) If f is a differentiable function on the interval $(0, 1)$ such that $f(x)f'(x) = 0$ for every value of x, then f must be a constant function.
   b) If $f : (0, 1) \to (0, 1)$ and $g : (0, 1) \to (0, 1)$ are two uniformly continuous functions, then the composite function $f \circ g$ must be uniformly continuous.

6. A function f taking values in the real numbers is called *upper semicontinuous* at a point x_0 of its domain if $\limsup_{x \to x_0} f(x) \le f(x_0)$.

   (Recall that lim sup denotes the largest limit that can be obtained along some sequence. One example of an upper semicontinuous function that fails to be continuous is the function that equals 0 when $x \neq x_0$ and equals 1 when $x = x_0$.)

   Prove that if f is upper semicontinuous at every point of a closed, bounded interval $[a, b]$, then f is necessarily bounded above on the interval.