Advanced Calculus I

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "give an example" requires a supporting explanation. In all problems, you should explain your reasoning in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B, and optionally the extra-credit question 10 in Section D.

Students in Section 200 (the honors section) should answer questions 1–3 in Part A and questions 7–9 in Part C, and optionally the extra-credit question 10 in Section D.

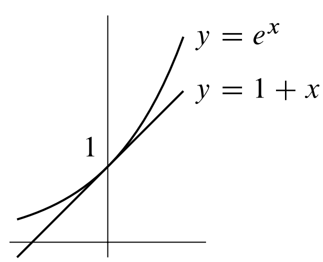
Part A, for both Section 200 and Section 501

In this part of the exam, your task is to analyze the following proposition from three different points of view.

Proposition. *If n is a natural number, then $e^n > 1 + n$.*

1. Prove the proposition by induction on n. [Recall that $e \approx 2.718$.]

2. The figure below suggests the more general statement that $e^x > 1 + x$ for every nonzero real number x (because the graph of e^x is convex).



   Prove that $e^x > 1 + x$ for every nonzero real number x by writing the Taylor polynomial for e^x of degree 1 and examining the remainder term.

   [Recall Lagrange's theorem about approximation by Taylor polynomials: $f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{1}{(n+1)!} f^{(n+1)}(c)(x - x_0)^{n+1}$ for some point c between x_0 and x. Take x_0 equal to 0 and n equal to 1.]

3. Evidently $e^x > 1$ for every positive real number x. By integrating on a suitable interval, deduce that $e^x > 1 + x$ for every positive real number x.

Advanced Calculus I

Part B, for Section 501 only

4. State the following theorems:
   (a) the Bolzano–Weierstrass theorem about sequences;
   (b) the squeeze theorem (sandwich theorem) about limits of functions;
   (c) Rolle's theorem about differentiable functions.

5. Give an example of a function f that satisfies all three of the following properties: the function f is continuous on the open interval $(0, 2)$, the function f is not differentiable at the point 1, and $\int_0^2 f(t)\,dt$ is a divergent improper integral.

6. Prove that $\displaystyle\lim_{n\to\infty} \frac{e^{\cos(n)}}{n} = 0.$

Part C, for Section 200 only

7. State the following theorems:
   (a) the Heine–Borel theorem characterizing compact sets of real numbers;
   (b) some theorem from this course in which the word "countable" appears;
   (c) some (other) theorem from this course named after Cantor.

8. Give an example of a function f that satisfies all of the following properties: the function f is continuous on the open interval $(0, 1)$, the upper limit $\limsup_{x\to 0+} f(x)$ equals ∞, the lower limit $\liminf_{x\to 0+} f(x)$ equals 0, and the integral $\int_0^1 f(t)\,dt$ is a convergent improper integral.

9. Prove that $\displaystyle\lim_{n\to\infty} \int_n^{n+1} \frac{t}{1+t^3}\,dt = 0.$

Advanced Calculus I

Part D, optional extra-credit question for both Section 200 and Section 501

10. Alfie, Beth, Gemma, and Delma are studying the limit

$$\lim_{x\to\infty}\left(\sqrt{x^2+9x}-\sqrt{x^2+4x}\right).$$

Alfie says, "simplifying the square roots converts the problem to

$$\lim_{x\to\infty}\left[(x+3\sqrt{x})-(x+2\sqrt{x})\right]=\lim_{x\to\infty}\sqrt{x}=\infty."$$

Beth says, "the limit of a sum is the sum of the limits, so the answer is

$$\lim_{x\to\infty}\sqrt{x^2+9x}-\lim_{x\to\infty}\sqrt{x^2+4x}=\infty-\infty=0."$$

Gemma says, "by the mean-value theorem, $\sqrt{u}-\sqrt{v}=\frac{1}{2\sqrt{c}}(u-v)$, so the limit equals

$$\lim_{x\to\infty}\frac{1}{2\sqrt{c}}\cdot[(x^2+9x)-(x^2+4x)]=\lim_{x\to\infty}\frac{1}{2\sqrt{c}}\cdot 5x=\infty."$$

Delma says, "by l'Hôpital's rule, the limit equals

$$\lim_{x\to\infty}\left(\frac{2x+9}{2\sqrt{x^2+9x}}-\frac{2x+4}{2\sqrt{x^2+4x}}\right)=\lim_{x\to\infty}\left(\frac{2x+9}{2x\sqrt{1+\frac{9}{x}}}-\frac{2x+4}{2x\sqrt{1+\frac{4}{x}}}\right)$$
$$=1-1=0."$$

Who (if anyone) is right, and why? Identify the mistakes in the erroneous arguments.