Examination 1

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. State one of the following: the Archimedean property of the real numbers; the Bolzano–Weierstrass theorem; Cauchy's criterion for convergence.

   Solution. See Theorems 1.11, 2.40, and 2.41 in the textbook.

2. Suppose A and B are bounded, non-empty sets of real numbers, and let C denote the union $A \cup B$. Show that $\sup C$ equals the maximum of the two numbers $\sup A$ and $\sup B$.

   Solution. An upper bound for a set is certainly an upper bound for every subset, so $\sup C$, the least upper bound of C, is an upper bound for both A and B. Therefore $\sup C$ is at least as big as both $\sup A$ (the least upper bound of A) and $\sup B$ (the least upper bound of B). In other words, $\sup C \geq \max\{\sup A, \sup B\}$.

   On the other hand, an arbitrary element c of C either is an element of the set A, in which case

   $$c \leq \sup A \leq \max\{\sup A, \sup B\},$$

   or is an element of the set B, in which case

   $$c \leq \sup B \leq \max\{\sup A, \sup B\}.$$

   In both cases, $c \leq \max\{\sup A, \sup B\}$. In other words, $\max\{\sup A, \sup B\}$ is an upper bound for C, so $\sup C \leq \max\{\sup A, \sup B\}$.

   What has been shown is that $\sup C$ is both greater than or equal to $\max\{\sup A, \sup B\}$ and less than or equal to the same quantity. Therefore equality holds.

3. Give an example of a set having at least one boundary point that is not an accumulation point and also at least one accumulation point that is not a boundary point. Explain why your example has the required properties.

   Solution. A boundary point that is not an accumulation point is an isolated point. An accumulation point that is not a boundary point is an interior point. Accordingly, any set that has both an isolated point and non-empty interior serves as an example. For instance, the set $(0, 1) \cup \{2\}$ is one concrete example.

4. Determine the smallest natural number k with the property that

   $$0.999 < \frac{n^2 - 1}{n^2 + 1} < 1.001 \qquad \text{for every natural number } n \text{ exceeding } 10^k.$$

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Solution. This problem is related to the definition of limit but asks both less and more. The problem asks for a cut-off value of n not for an arbitrary value of ε but only for one specific value of ε, which is less than the definition of limit requires. On the other hand, the problem asks for an optimal cut-off, which is more than the definition of limit requires.

The following argument shows that the minimal value of k is 2. In other words, the double inequality holds when $n > 10^2$, but there is at least one value of n between 10^1 and 10^2 for which the inequality fails.

Since $n^2 - 1 < n^2 + 1$, the fraction $\dfrac{n^2-1}{n^2+1}$ is always less than 1, so the right-hand part of the required inequality holds for *every* natural number n. Accordingly, only the left-hand inequality needs to be studied. Subtracting 1 from both sides produces an equivalent inequality: namely,

$$-0.001 < \frac{n^2-1}{n^2+1} - 1 = \frac{-2}{n^2+1}.$$

Multiplying by -1 reverses the direction of the inequality and yields another equivalent inequality:

$$0.001 > \frac{2}{n^2+1}. \tag{*}$$

Now if $n > 10^2$, that is, $n > 100$, then

$$\frac{2}{n^2+1} < \frac{2}{100^2} = 0.0002 < 0.001,$$

so (*) holds. On the other hand, if $n = 11$, which is a value between 10^1 and 10^2, then

$$\frac{2}{n^2+1} = \frac{2}{122} = \frac{1}{61} > \frac{1}{100} = 0.01 > 0.001,$$

so (*) fails. In other words, the optimal cut-off for n of the form 10^k is 10^2.

Remark You can even solve inequality (*) explicitly. An equivalent statement is that $n^2 + 1 > 2000$, or $n^2 > 1999$. If you have a calculator at hand, then you can punch some buttons to see that the cut-off value for n is between 44 and 45. But even without a calculator, you can deduce that the cut-off is between 40 and 50, since $40^2 = 1600 < 1999$ and $50^2 = 2500 > 1999$. Pursuing this idea, observe from the binomial expansion that $(40 + k)^2 = 40^2 + 2 \cdot 40 \cdot k + k^2$. Therefore $44^2 = 40^2 + 2 \cdot 40 \cdot 4 + 4^2 = 1936$, and $45^2 = 40^2 + 2 \cdot 40 \cdot 5 + 5^2 = 2025$. Thus no technological assistance is needed to see that the inequality holds when $n \geq 45$ and fails when $n \leq 44$.

5. Suppose E is a compact set of real numbers and F is a closed set. Is the intersection $E \cap F$ necessarily compact? Give either a proof or a counterexample, as appropriate.

Solution. Use the theorem that a set of real numbers is compact if and only if the set is simultaneously closed and bounded.

Since E is bounded, so is every subset of E. Since $E \cap F$ is a subset of E, the set $E \cap F$ is bounded. Moreover, the compact set E is closed, and F is closed, and the intersection of two closed sets is closed, so $E \cap F$ is closed. Thus the intersection $E \cap F$ is both bounded and closed, hence compact.

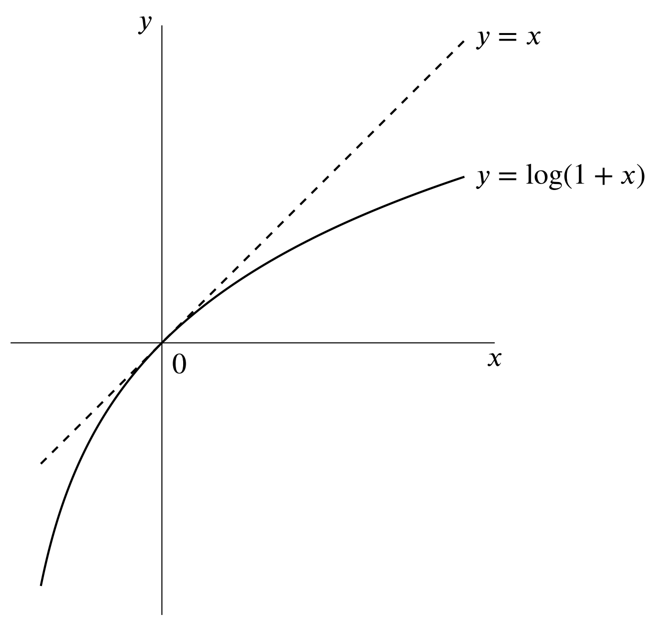
Remark On the other hand, the union $E \cup F$ need not be compact: this set is necessarily closed but could be unbounded. For a specific counterexample, take E to be the singleton set $\{0\}$ and F to be the set of natural numbers; then E is compact (closed and bounded) and F is closed (contains all its accumulation points because there are none), and the union $E \cup F$ is not compact (because not bounded).

6. Consider the sequence defined recursively as follows:

$$x_1 = 1, \qquad \text{and} \qquad x_{n+1} = \log(1 + x_n) \qquad \text{when } n \geq 1.$$

Here "log" means the natural logarithm function (which is often called "ln" in elementary mathematics). Say as much as you can about the value of $\limsup_{n\to\infty} x_n$ for this sequence.

Hint: Use the following diagram, which shows that the expression $\log(1+x)$ is an increasing function of x whose graph is concave down. The tangent line at the origin has slope 1.



Solution. First observe that $\log(1 + x)$ is positive when x is positive (either by looking at the diagram or by invoking prior knowledge about the logarithm function). Therefore if

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$x_n > 0$ for a certain value of n, then $x_{n+1} > 0$ too. Since $x_1 > 0$ (basis step), induction implies that $x_n > 0$ for every natural number n.

The diagram also shows that $\log(1 + x) < x$ when x is positive. Accordingly, $x_{n+1} = \log(1 + x_n) < x_n$ for every natural number n. Thus the sequence is decreasing.

Since the sequence is both decreasing and bounded below by 0, the monotone convergence theorem implies that the sequence converges to the greatest lower bound, say L. And $L \geq 0$, since limits respect the weak order relation on the real numbers.

Knowing now that the sequence has a limit, you can deduce from the recursive definition of the sequence that $L = \log(1 + L)$. The diagram shows that the only nonnegative real number x for which $x = \log(1 + x)$ is 0. Thus $L = 0$.

Since the sequence converges, the value of $\limsup_{n\to\infty} x_n$ equals $\lim_{n\to\infty} x_n$. In conclusion, $\limsup_{n\to\infty} x_n = 0$.