# Final Examination

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B, and optionally the extra-credit question 10 in Part D.

Students in Section 200 should answer questions 1–3 in Part A and questions 7–9 in Part C, and optionally the extra-credit question 10 in Part D.

## Part A, for both Section 501 and Section 200

1. Consider a function $f$ defined as follows:

$$f(x) = 409x - x^{409} \qquad \text{when} \qquad 0 < x < 2.$$

Does the function $f$ attain a maximum on the open interval $(0, 2)$? What about a minimum? Why or why not?

**Solution.** The extreme-value theorem implies that the continuous function $f$ attains both a maximum and a minimum on the *closed* bounded interval $[0, 2]$. Accordingly, the question reduces to deciding whether an extreme value occurs at an endpoint.

Now $f'(x) = 409 - 409x^{408} = 409(1 - x^{408})$, so $f'(x) > 0$ when $0 < x < 1$, and $f'(x) < 0$ when $1 < x < 2$. Therefore the function $f$ increases on the interval $(0, 1)$ and decreases on the interval $(1, 2)$. Consequently, the function $f$ attains an absolute maximum on the interval $(0, 2)$ at the midpoint. On the other hand, the minimum on the interval $[0, 2]$ is taken at an endpoint (actually the right-hand endpoint): there is no minimum on the open interval $(0, 2)$.

2. Consider a sequence $\{x_n\}$ of real numbers defined recursively as follows:

$$x_1 = 1, \qquad \text{and} \qquad x_{n+1} = \frac{409 + x_n}{2} \quad \text{when} \quad n \geq 1.$$

Does $\lim_{n\to\infty} x_n$ exist? Explain why or why not.

**Solution. Method 1.** Prove by induction that the sequence is increasing and bounded above. The completeness property of the real numbers then implies that the sequence converges to the least upper bound.

To prove by induction that the sequence is bounded above by 409, first observe that the initial term $x_1$ is less than 409, so the basis step is valid. Supposing now for some natural number $k$ that $x_k < 409$, you can deduce that

$$x_{k+1} = \frac{409 + x_k}{2} < \frac{409 + 409}{2} = 409.$$

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Thus the hypothesis that $x_k < 409$ implies the conclusion that $x_{k+1} < 409$, so the induction step is valid.

There are two ways to prove that the sequence is increasing, equivalently, that $x_{n+1} - x_n > 0$ for every natural number $n$. One way is to observe that

$$x_{n+1} - x_n = \frac{409 + x_n}{2} - x_n = \frac{409 - x_n}{2},$$

and this expression is positive because $x_n < 409$ (by the previous step). Alternatively, you could prove by induction that the sequence is increasing, as follows. First observe that $x_2 = \frac{409+1}{2} = 205 > 1 = x_1$, so the basis step is valid. Supposing now for some natural number $k$ that $x_{k+1} > x_k$, you can deduce that

$$x_{k+2} = \frac{409 + x_{k+1}}{2} > \frac{409 + x_k}{2} = x_{k+1}.$$

Thus the hypothesis that $x_{k+1} > x_k$ implies the conclusion that $x_{k+2} > x_{k+1}$, so the induction step is valid.

In summary, mathematical induction implies that the sequence is bounded and increasing. The theorem on monotone convergence yields that the sequence converges. (The limit of the sequence turns out to be 409, which is the least upper bound of the sequence.)

**Method 2.** If the sequence does converge to a limit $L$, then $L$ must satisfy the following equation:

$$L = \frac{409 + L}{2}.$$

Therefore the only possible candidate for a limit $L$ is 409. Additional work is required to demonstrate that the sequence really does converge to this conjectural limit.

Showing that $x_n$ converges to 409 is equivalent to showing that $x_n - 409$ converges to zero, so studying this difference is a natural idea. Observe that

$$x_n - 409 = \frac{409 + x_{n-1}}{2} - 409 = \frac{1}{2}\left(x_{n-1} - 409\right) \qquad \text{when } n > 1.$$

A straightforward induction argument then shows that

$$x_n - 409 = \frac{1}{2^{n-1}}\left(x_1 - 409\right)$$

for every natural number $n$. But $x_1 = 1$, so

$$x_n = 409 - \frac{408}{2^{n-1}}.$$

Since $1/2^{n-1} \to 0$ when $n \to \infty$, this explicit formula for the general term $x_n$ confirms the conjecture that $x_n \to 409$.

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3. Consider the following three properties that a function $f$ might or might not have:
   (C) $f$ is continuous on the interval $[0,2]$.
   (D) $f$ is differentiable on the interval $[0,2]$.
   (I) $f$ is Riemann integrable on the interval $[0,2]$.

   There are six possible implications between these properties: (C) $\Rightarrow$ (D); (D) $\Rightarrow$ (C); (C) $\Rightarrow$ (I); (I) $\Rightarrow$ (C); (D) $\Rightarrow$ (I); (I) $\Rightarrow$ (D). Which implications are valid? Why? For the implications that are not valid, give counterexamples.

   **Solution.** A continuous function need not be differentiable; a standard counterexample is the absolute-value function $|x-1|$, a continuous function that fails to be differentiable when $x = 1$. On the other hand, a theorem says that every differentiable function is continuous.

   A theorem says that every continuous function is Riemann integrable. On the other hand, a step function (the ceiling function, for example) is Riemann integrable but not continuous.

   A differentiable function is continuous and hence Riemann integrable. On the other hand, the preceding example shows that an integrable function need not be continuous and hence need not be differentiable.

# Part B, for Section 501 only

4. State the following theorems:
   a) l'Hôpital's rule (some version);
   b) the fundamental theorem of calculus (relating derivatives and integrals);
   c) some (other) theorem from this course that involves the concept of compactness.

   **Solution.** Various versions of l'Hôpital's rule are stated in Section 7.11 in the textbook. The two halves of the fundamental theorem of calculus are statements 8.8 and 8.9 in section 8.3 of the textbook; more general versions for the Riemann integral are statements 8.26 and 8.27 in Section 8.7. Some theorems having to do with compactness are the Bolzano–Weierstrass theorem, the Heine–Borel theorem, the characterization of compact subsets of $\mathbb{R}$ as the sets that are both closed and bounded, and the extreme-value theorem (a continuous function on a compact set attains a maximum and a minimum).

5. Prove that $2^n \geq 2n$ for every natural number $n$.

   **Solution. Method 1.** The statement can be proved by induction. When $n = 1$, both sides of the inequality are equal to 2, so the basis step is valid. Supposing now that $2^k \geq 2k$ for some natural number $k$, you can deduce that

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k && \text{(by definition of exponentiation)} \\
&\geq 2 \cdot 2k && \text{(by the induction hypothesis)} \\
&= 2k + 2k && \text{(by definition of multiplication)} \\
&\geq 2k + 2 && \text{(because } k \geq 1\text{)} \\
&= 2(k+1).
\end{aligned}$$

Thus the hypothesis that $2^k \geq 2k$ implies the conclusion that $2^{k+1} \geq 2(k+1)$, so the induction step is valid. Therefore mathematical induction implies that the required inequality holds for every natural number.

**Method 2.** The inequality evidently holds (actually with equality) when $n = 1$ and when $n = 2$. Calculus techniques can be used as follows to show that $2^x$ exceeds $2x$ when the real number $x$ exceeds 2, a conclusion even stronger that what is required.

Let $f(x)$ denote $2^x$. The points $(1, 2)$ and $(2, 4)$ on the graph of $f$ are joined by a chord of slope 2. The mean-value theorem implies that $f'(c) = 2$ for some value of $c$ between 1 and 2. Since the exponential function $f$ is convex (that is, has increasing derivative), the value of $f'(x)$ exceeds 2 when $x$ exceeds 2. Consequently, the function $2^x - 2x$ has positive derivative when $x > 2$, and this function has value 0 when $x = 2$. Thus when $x \geq 2$, the function $2^x - 2x$ increases starting from the value 0, so $2^x - 2x > 0$ when $x > 2$.

Natural numbers are, in particular, real numbers, so the preceding conclusion implies that $2^n > 2n$ when $n > 2$.

6. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$, yet $\displaystyle\lim_{x\to\infty} \int_1^x f(t)\,dt = \infty$.

**Solution.** Here is one example:

$$f(x) = \begin{cases} 1, & \text{when } x < 1, \\ \frac{1}{x}, & \text{when } x \geq 1. \end{cases}$$

Evidently, $\lim_{x\to\infty} f(x) = 0$. Moreover,

$$\int_1^x f(t)\,dt = \int_1^x \frac{1}{t}\,dt = \log(x) - \log(1) = \log(x),$$

and $\lim_{x\to\infty} \log(x) = \infty$.

# Part C, for Section 200 only

7. State the following theorems:
   a) some theorem from this course named after a European mathematician whose name begins with the letter "C";
   b) some (other) theorem from this course that involves the concept of a covering by open sets or by closed sets;
   c) some (other) theorem from this course that involves the intermediate-value property (Darboux property).

   **Solution.** Some "C" theorems are Cantor's theorem about uncountability of the real numbers, Cantor's theorem about nested sets, Cauchy's criterion for convergence, Cauchy's mean-value theorem, Cauchy's construction of an integral, and Cousin's covering lemma.

   Some theorems involving coverings are Cousin's lemma, the Heine–Borel theorem (which characterizes compactness through open covers), and Lebesgue's characterization of the functions that are Riemann integrable (the concept of measure zero involves a covering by open intervals).

   Two theorems about the intermediate-value property are that every continuous function has the property and that every derivative has the property.

8. Prove that $2^n \ge n^2$ for every natural number $n$ larger than 3.

   **Solution. Method 1.** The statement can be proved by induction. Evidently, equality holds when $n = 4$, which is the basis step. Suppose now that $2^k \ge k^2$ for some natural number $k$ larger than 3. Then

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k && \text{(by definition of exponentiation)}\\
&\ge 2 \cdot k^2 && \text{(by the induction hypothesis)}\\
&= k^2 + k^2 && \text{(by definition of multiplication)}\\
&> k^2 + 3k && \text{(because } k > 3\text{)}\\
&= k^2 + 2k + k && \text{(by definition of multiplication)}\\
&> k^2 + 2k + 1 && \text{(because } k > 1\text{)}\\
&= (k+1)^2.
\end{aligned}$$

   Thus the hypothesis that $2^k \ge k^2$ implies the conclusion that $2^{k+1} \ge (k+1)^2$, which is the induction step. Therefore mathematical induction implies that the required inequality holds for every natural number larger than 3.

   **Method 2.** Calculus techniques can be used as follows to show that $2^x$ exceeds $x^2$ when the real number $x$ exceeds 4, a conclusion even stronger that what is required. (Equality evidently holds when $x = 4$.)

To say that $2^x > x^2$ when $x > 4$ is equivalent to saying that $x \log 2 > 2 \log x$ when $x > 4$ (because the natural logarithm function is increasing). Let $f(x)$ denote $x \log 2 - 2 \log x$ when $x > 0$. Then $f'(x) = \log 2 - \frac{2}{x}$, so $f'(x) > \log 2 - \frac{1}{2}$ when $x > 4$. If you have in your head that $\log 2$ is about 0.69, then you can see immediately that $\log 2 - \frac{1}{2} > 0$. Otherwise, you can say that $\log 2 - \frac{1}{2} = \frac{1}{2}(2 \log 2 - 1) = \frac{1}{2}(\log 4 - 1)$, and now you can see that this expression is positive because $\log 4 > 1$ (since $4 > e$). Since the derivative $f'(x)$ is positive when $x > 4$, the function $f(x)$ is increasing when $x > 4$. But $f(4) = 0$, so $f(x) > 0$ when $x > 4$.

Natural numbers are, in particular, real numbers, so the preceding conclusion implies that $2^n > n^2$ when $n > 4$.

9. Give an example of a bounded function $f$ on the interval $[0, 1]$ such that $\lim_{x\to 0+} f(x)$ does not exist, yet $\lim_{x\to 0+} \int_x^1 f(t)\,dt$ does exist.

**Solution.** One concrete example is the (discontinuous) function that is equal to 1 when $x$ is the reciprocal of a natural number and equal to 0 otherwise. Evidently $\limsup_{x\to 0+} f(x) = 1$, and $\liminf_{x\to 0+} f(x) = 0$, so $\lim_{x\to 0+} f(x)$ does not exist. When $x > 0$, the function $f$ has only a finite number of discontinuities in the interval $[x, 1]$, so the Riemann integral $\int_x^1 f(t)\,dt$ certainly exists. The lower sum for an arbitrary partition is equal to 0, so the existence of the integral implies that the value of the integral must be 0. Therefore $\lim_{x\to 0+} \int_x^1 f(t)\,dt$ exists and equals 0.

There are many other examples, such as $\sin(1/x)$. In fact, if $f$ is an arbitrary bounded function that is Riemann integrable on the interval $[0, 1]$, then $\lim_{x\to 0+} \int_x^1 f(t)\,dt$ exists and equals $\int_0^1 f(t)\,dt$. The reason is that

$$\left| \int_0^1 f(t)\,dt - \int_x^1 f(t)\,dt \right| = \left| \int_0^x f(t)\,dt \right|,$$

and the right-hand side is bounded above by $x$ times an upper bound for $|f|$, hence tends to 0 when $x$ tends to 0. Therefore an arbitrary bounded function that is Riemann integrable on $[0, 1]$ but discontinuous from the right at 0 serves as an example.

## Part D, optional extra-credit question for both Section 200 and Section 501

10. Write an essay on the following topic: What is the most important concept or principle or theorem from this course? Why?

**Solution.** There is no single right answer to this question. The goal is to make a reasoned argument with supporting evidence to justify your choice.