Recap from last time

The real numbers are characterized by being a complete, ordered field.

Complete means that every non-empty subset that is bounded above has a least upper bound.

Supremum is a synonym for least upper bound.

Infimum is a synonym for greatest lower bound.

You can go back and forth between infimum and supremum by observing that $\inf(S) = -\sup(-S)$, where $-S$ means the set of negatives of all the elements of the set S.

If $\sup(S)$ is an element of the set S, you are allowed to write $\max(S)$. Similarly for inf and min.

A consequence of completeness

Theorem (Archimedean property of $\mathbb{R}$)

If x and y are two arbitrary positive real numbers, then there exists a natural number n such that $nx > y$.

Proof.

Seeking a contradiction, suppose for some x and y no such n exists. That is, $nx \leq y$ for every positive integer n. Then dividing by the positive number x shows that $n \leq y/x$ for every positive integer n.

Since the natural numbers have an upper bound y/x, there is by completeness a least upper bound, say s.

When n is a natural number, $n \leq s$; but $n + 1$ is a natural number too, so $n + 1 \leq s$. Add -1 to both sides to deduce that $n \leq s - 1$ for every natural number n.

Then $s - 1$ is an upper bound for the natural numbers that is smaller than the supposed least upper bound s. Contradiction. $\square$

Density of $\mathbb{Q}$ in $\mathbb{R}$

If $x < y$, then there exists a rational number between x and y.
Why?
By Archimedean property, there is some positive integer n such that $n(y - x) > 1$. If we can show that the interval (nx, ny) contains some integer k, then $nx < k < ny$, so dividing by the positive integer n shows that $x < k/n < y$, so k/n is the required rational number between x and y.
The set of integers that are less than or equal to nx is bounded above, so has a supremum, and this supremum is a maximum (is in the set); see A.4.10 in the Appendix. Call it m.
Then $nx < m + 1$ by definition of m. Also $m \leq nx$, so $m + 1 \leq nx + 1 < nx + n(y - x) = ny$. So $m + 1$ is the required integer k. □

Assignment to hand in next time

Exercise 2 on page 18 in Section 2.2: namely, show that

$$\bigcap_{n=1}^{\infty} (0, y/n] = \varnothing$$

for every positive real number y.