Existence of square roots

Theorem

Every positive real number has a square root. More precisely, if c is a positive real number, then there exists one and only one positive real number x such that $x^2 = c$.

Proof.

Say $S = \{ x \in \mathbb{R} : x \text{ is positive and } x^2 \geq c \}$, and $g = \inf(S)$. The goal is to show $g^2 = c$.
The plan is to show that if $g^2 < c$, a contradiction arises; and if $g^2 > c$, a contradiction arises. □

Suppose $g^2 < c$. What contradiction arises?

The Archimedean property implies the existence of a positive integer n for which $\frac{2g+1}{n} < c - g^2$.
[How did I know to write this inequality? I worked out a side calculation.]
Then

$$\left(g + \frac{1}{n}\right)^2 = g^2 + \frac{2}{n}g + \frac{1}{n^2} \leq g^2 + \frac{2}{n}g + \frac{1}{n} < g^2 + (c - g^2) = c.$$

Now if $x \in S$, then $x^2 \geq c$, so the preceding inequality implies that $x^2 \geq (g + \frac{1}{n})^2$. By the lemma from last time, $x \geq g + \frac{1}{n}$.
Therefore $g + \frac{1}{n}$ is a lower bound for the set S, a greater lower bound than g. Contradiction.

Suppose $g^2 > c$. What contradiction arises?

By the Archimedean principle, there is a positive integer n such that $g > 1/n$ and also

$$g^2 - c > \frac{2}{n}g > \frac{2}{n}g - \frac{1}{n^2}.$$

Therefore $(g - \frac{1}{n})^2 = g^2 - \frac{2}{n}g + \frac{1}{n^2} > c$, so the positive number $g - \frac{1}{n}$ is an element of the set S, contradicting that g is a lower bound for S.

Conclusion of the proof

By the trichotomy law, it must be that $g^2 = c$. Accordingly, the existence of square roots is proved.

What about uniqueness? If $g_1^2 = c$ and $g_2^2 = c$, then

$$0 = g_1^2 - g_2^2 = (g_1 - g_2)(g_1 + g_2).$$

Dividing by the positive quantity $g_1 + g_2$ shows that $0 = g_1 - g_2$, that is, $g_1 = g_2$.

Introduction to sequences

Nobody is in doubt that $\lim_{n\to\infty} \frac{1}{n} = 0$.

But what about

$$\lim_{n\to\infty} \frac{n!\, e^n}{n^{n+\frac{1}{2}}} = \sqrt{2\pi}$$

(Stirling's formula).

What is a sequence?

Examples:

- $(-1)^n$, where n represents a positive integer.
- 3, 1, 4, 1, 5, 9, 2, 6, 5, ... (digits of π).
- 1, 2, 3, ... (positive integers)
- 3, 3, 3, ... (a constant sequence)

A *sequence* is a list, and a *series* is the sum of the elements of a list.

A sequence is a function whose domain is the set of natural numbers.

How to denote a sequence? You could write $f(n)$ or f_n or (f_n) or $\{f_n\}$ or $(x_n)_{n \geq 1}$.

Special types of sequences of real numbers

- A sequence (x_n) is bounded above if there exists a real number b such that $x_n \leq b$ for every natural number n.
- A sequence (x_n) is not bounded above if for every real number b there exists a natural number n such that $x_n > b$.

to be continued . . .

Assignment to turn in next time

- Exercise 1 on page 29. [Bijective means one-to-one correspondence: see page 226 in the Appendix.]
- Exercise 1 on pages 31–32.