Some examples of sequences of real numbers

1. $(\cos(n\pi))_{n\geq 0}$
2. $(n^2)_{n\geq 0}$
3. $(1/n)_{n\geq 1}$

Which of these examples is/are bounded above?
Sequences 1 and 3 are bounded above, with supremum equal to 1.
bounded below?
Sequences 2 and 3 have 0 as greatest lower bound; sequence 1 has -1 as greatest lower bound.
bounded?
Sequences 1 and 3.
increasing?
Sequence 2 is strictly increasing.
decreasing?
Sequence 3 is strictly decreasing.
What about the constant sequence 2, 2, 2, ...?
This sequence is both (weakly) increasing and (weakly) decreasing.

Montonic means either increasing or decreasing. Examples 2 and 3 are monotonic. A constant sequence is monotonic.

Terminology: ultimately, frequently

A sequence (x_n) is *ultimately* (or *eventually*) in a set S if all but a finite number of terms are elements of S. In symbols: $\exists N \in \mathbb{N}$ such that $x_n \in S$ when $n \geq N$.
Which of the examples is/are ultimately positive?
Sequence 1 is not, but sequences 2 and 3 are ultimately positive.

A sequence (x_n) is *frequently* (or *infinitely often*) in a set S if $\forall N \; \exists n > N$ such that $x_n \in S$.
Which of the examples is/are frequently positive?
All three are frequently positive. Sequence 1 is also frequently negative.
Equivalences:

- not frequently in $S \iff$ ultimately not in S
- not ultimately in $S \iff$ frequently not in S

Null sequences in the real numbers

A sequence (x_n) is *null* if for every open interval containing 0, the sequence is ultimately in that interval.
In symbols: $\forall\varepsilon > 0\ \exists N$ such that $|x_n| < \varepsilon$ when $n \geq N$.
Negation: (x_n) is not a null sequence means, in symbols, $\exists\varepsilon > 0\ \forall N\ \exists n \geq N$ such that $|x_n| \geq \varepsilon$.

Assignment to hand in next time

- Exercise 4 on page 34 in Section 3.1.
- Exercise 4 on page 39 in Section 3.3.