Null sequences and limits

Recap: A sequence (x_n) is a *null sequence* if for every open interval containing 0, the sequence is ultimately in that interval.
In symbols: $\forall \varepsilon > 0 \ \exists N$ such that $|x_n| < \varepsilon$ when $n \geq N$.

Definition

A sequence $(x_n)_{n\geq 1}$ of real numbers *converges* to a limit L if the sequence $(x_n - L)_{n\geq 1}$ is a null sequence.

Notation: $\lim_{n\to\infty} x_n = L$, or simply $x_n \to L$.

Example: $\frac{1}{n} \to 0$. Why? Given a positive ε, could take the cut-off N to be $\frac{1}{\varepsilon}$ or $\left\lceil \frac{1}{\varepsilon} \right\rceil$.

Nonexample: The sequence (n) fails to converge to any limit, that is, there is no number L for which $(n - L)$ is a null sequence.

Necessary condition for convergence

A convergent sequence must be bounded.
Indeed, if $x_n \to L$, then taking ε equal to 2 (for instance) in the definition of limit shows that there exists some N such that if $n \geq N$, then $|x_n - L| < 2$, or $-2 < x_n - L < 2$, or $L - 2 < x_n < L + 2$.
An upper bound for all the terms of the sequence is $\max\{x_1, x_2, \ldots, x_N, L + 2\}$.
Similarly, a lower bound for the sequence is $\min\{x_1, x_2, \ldots, x_N, L - 2\}$.
Example: $(\sin(n\pi/2))$ is bounded between -1 and 1 but fails to converge.
Boundedness is a necessary but not sufficient condition for convergence.

Bounded monotonic sequences of real numbers converge

Theorem

If a sequence of real numbers is (i) increasing and (ii) bounded above, then the sequence converges.
The limit is the supremum of the sequence.

Proof.

Suppose (x_n) is a sequence satisfying the hypotheses, and L denotes the supremum. Let ε be an arbitrary positive number. The number $L - \varepsilon$ is smaller than L, hence is not an upper bound for the sequence, so there is some natural number N for which $L - \varepsilon < x_N \leq L$.
Since the sequence is increasing, if $n > N$, then $x_n \geq x_N$. Therefore $L - \varepsilon < x_n \leq L$ when $n \geq N$, that is, $|x_n - L| < \varepsilon$ when $n \geq N$. □

Example

Suppose $x_1 = 1$ and $x_{n+1} = \sqrt{2x_n + 3}$ when n is a positive integer. Discuss convergence.

First claim: the sequence is bounded above by 3.
Proof by induction. Evidently $x_1 < 3$, so the basis step holds.
Induction step. Suppose $x_k \leq 3$ for a certain positive integer. The goal is to deduce that $x_{k+1} \leq 3$. By the recursive definition, $x_{k+1} = \sqrt{2x_k + 3} \leq \sqrt{2 \cdot 3 + 3} = \sqrt{9} = 3$. By induction, all terms of the sequence are less than or equal to 3.
Second claim: the sequence is increasing. Indeed, $x_{n+1} = \sqrt{2x_n + 3} \geq \sqrt{2x_n + x_n} = \sqrt{3x_n} \geq \sqrt{x_n \cdot x_n} = x_n$ (because $3 \geq x_n$). Since this inequality holds for a general n, the sequence is increasing.
By the previous theorem, this sequence converges (and the value of the limit is least upper bound).

Assignment to hand in next time

Exercise 10 on page 43.