Interaction of limits with operations on $\mathbb{R}$

Theorem 3.4.8 says that limits are compatible with the field operations and with the order relation $\leq$.

Example: The strict order relation $<$ is not necessarily preserved by taking limits. If $a_n = 1 - \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$, then $a_n < b_n$ (strict inequality) for every n, but $\lim_{n\to\infty} a_n = 1 = \lim_{n\to\infty} b_n$.

Remark on the assignment

Using the definition of limit, we need to address the inequality $\frac{n!}{n^n} < \varepsilon$. How big must n be to make such an inequality hold?

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n} \leq \frac{1}{n}$$

So if N is chosen to be $1/\varepsilon$, then if $n \geq N$, we can deduce that $1/n \leq \varepsilon$, so $0 \leq n!/n^n \leq \varepsilon$ too.

Sandwich theorem (squeeze theorem)

Theorem

Suppose $x_n \leq y_n \leq z_n$ for every n. If $x_n \to L$ and $z_n \to L$ (the same limit L), then $y_n \to L$. (The limit exists and equals L.)

Proof.

By hypothesis, $(x_n - L)$ is a null sequence, and $(z_n - L)$ is a null sequence, and $x_n - L \leq y_n - L \leq z_n - L$ for every n. An interval that contains the numbers $x_n - L$ and $z_n - L$ contains all the numbers in between, hence contains the number $y_n - L$.
Then the definition of null sequence implies that $(y_n - L)$ is a null sequence too.

□

Subsequences

Example: $x_n = (-1)^n + \frac{1}{n}$
$x_{2n} \to 1$ and $x_{2n+1} \to -1$, so the sequence does not have a limit, but there are two *sub*sequences that have limits.
The largest limit of any convergent subsequence of a sequence (x_n) is called the limit superior, abbreviated $\limsup_{n\to\infty} x_n$. In the example above, $\limsup x_n = 1$.
The smallest limit of any convergent subsequence is the limit inferior, abbreviated $\liminf$. In the example, $\liminf x_n = -1$.

Assignment to hand in next time

Exercise 7 on page 55 in Section 3.7.