Recap from last time

If $x \in E$, then

- x is an interior point of E if E is a neighborhood of x (that is, contains an open interval that contains x);
- x is an isolated point of E if some neighborhood of x contains no other point of E.

If $x \in \mathbb{R}$ (not necessarily in E), then

- x is a boundary point of E if every neighborhood of x intersects both E and the complement of E;
- x is a limit point of E if every neighborhood of x contains some points of E different from x. In other words, E contains a sequence of points different from x that converges to x.

Remark on notation

When E is a subset of $\mathbb{R}$, the *complement* of E is $\{x \in \mathbb{R} : x \notin E\}$.

Notations for the complement of E:

- $\complement E$ (the book's notation)
- E^c (do not confuse the superscript with an exponent)
- $\mathbb{R} \setminus E$ or $\mathbb{R} - E$
- $\overline{E}$ or E' (used by some authors, but in our book, $\overline{E}$ means the closure of E, and E' means the set of limit points of E)

Open sets and closed sets

Warning! In mathematics, the words “open” and “closed” are not opposites. A set can be both open and closed at the same time; or neither open nor closed.

A set is *open* when it is a neighborhood of each of its points.

A set is *closed* when the complement is open.

Example. $\mathbb{Z}$ (the integers) is a closed subset of $\mathbb{R}$ because $\mathbb{R} - \mathbb{Z}$ is a union of open intervals.
Example. $\varnothing$ is open (by default) and closed because $\mathbb{R}$ is open.
Similarly, $\mathbb{R}$ is both open and closed.
Example. $\mathbb{Q}$ (the rational numbers) is neither open nor closed, because the set contains no intervals, and the complement contains no intervals.

Characterizations of closed sets

The following properties of a subset E of $\mathbb{R}$ are equivalent:

1. E is closed.
2. The complement of E is open.
3. E contains all its boundary points.
4. E contains all its limit points.
5. For every sequence (x_n), if $x_n \in E$ for every n, and if the sequence (x_n) converges to a limit L, then $L \in E$.

Important remark

Not closed is different from open.
Not open is different from closed.
The set $[2, 5)$ is not closed, but also is not open.

Interior and closure

From last time: the *interior* of a set E is the set of all interior points of E (in other words, the largest open subset of E). Notation: E° or $\mathring{E}$ or $\mathrm{Int}(E)$.

The *closure* of a set E is the union of E and the set of limit points of E (in other words, the smallest closed superset of E). Notation: $\overline{E}$ or $\mathrm{Cl}(E)$.

If E is open, then the interior of E equals E.
If E is closed, then the closure of E equals E.

Exercise

For each of the following sets, identify the interior of the set and the closure of the set.

- $\{1/2, 1/3, 1/4, \ldots\} = \{1/n : n \in \mathbb{N}, n \geq 2\}$
  Answer: interior is empty, closure is $\{0\} \cup \{1/2, 1/3, 1/4, \ldots\}$.
- $\{0\} \cup \{1/2, 1/3, 1/4, \ldots\}$
  Answer: interior is empty, closure is the set itself.
- $\mathbb{R} \setminus \mathbb{Z}$
  Answer: interior equals the set, closure is $\mathbb{R}$.
- $\mathbb{R} \setminus \mathbb{Q}$
  Answer: interior is empty, closure is $\mathbb{R}$.
- $\{x \in \mathbb{R} : x^2 < 2\}$
  Answer: interior is the set itself, closure is $[-\sqrt{2}, \sqrt{2}\,]$.
- $\{x \in \mathbb{Q} : x^2 < 2\}$
  Answer: interior is empty, closure is $[-\sqrt{2}, \sqrt{2}\,]$.

Exercise

How do the operations of taking intersection and union interact with interior and closure? Namely, resolve the following questions when A and B are arbitrary sets.
(Notation: A° is the interior of A, and $\overline{A}$ is the closure of A.)

1. Are the sets $(A \cap B)^\circ$ and $A^\circ \cap B^\circ$ always equal?
   If not, is one always a subset of the other?
2. Are the sets $(A \cup B)^\circ$ and $A^\circ \cup B^\circ$ always equal?
   If not, is one always a subset of the other?
3. Are the sets $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ always equal?
   If not, is one always a subset of the other?
4. Are the sets $\overline{A \cup B}$ and $\overline{A} \cup \overline{B}$ always equal?
   If not, is one always a subset of the other?