Announcement

- No office hour on March 31 (Friday) or April 3 (Monday).
  I will be away from campus.

Exercise on quantifiers

Do quantifiers commute?

- Is $\forall a \, \exists d$ the same as $\exists d \, \forall a$?
  For every Aggie there exists a day when the Aggie says Howdy.
  versus
  There exists a day when every Aggie says Howdy.
  not the same meaning
- Is $\forall a \, \forall d$ the same as $\forall d \, \forall a$?
  the same meaning
- Is $\exists a \, \exists d$ the same as $\exists d \, \exists a$?
  the same meaning

Continuity versus uniform continuity

Continuity of f at every point of a set.
$\forall c\ \forall \varepsilon\ \exists \delta\ \forall x$: $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
[δ may depend on both ε and c]

Uniform continuity of f on a set.
$\forall \varepsilon\ \exists \delta\ \forall c\ \forall x$: $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
[δ depends only on ε]

Example

$f : \mathbb{R} \to \mathbb{R}$

- $f(x) = x$ is a uniformly continuous function (we can take δ equal to ε)
- $f(x) = x^2$ is continuous at each point but not uniformly continuous.

Why is x^2 not uniformly continuous?

$|f(x) - f(c)| = |x^2 - c^2| = |x - c|\,|x + c| \approx |x - c| \cdot 2|c|$ when x is close to c.

To make $|f(x) - f(c)|$ less than a prescribed ε, need $|x - c|$ less than approximately $\varepsilon/(2|c|)$. So $\delta \approx \varepsilon/(2|c|)$, which depends on both ε and c.

The function is continuous at each point, but not uniformly continuous on the whole domain.

A magic theorem for compact sets: Theorem 6.6.1

Theorem

If f is continuous at every point of a **compact** *set, then f is automatically uniformly continuous on the set.*

Proof using the Heine–Borel covering property.

Fix a target positive ε. For each point c in the set, continuity at c implies the existence of a positive δ_c such that if x is in the set and $|x - c| < \delta_c$, then $|f(x) - f(c)| < \frac{1}{2}\varepsilon$.
Consider the open intervals $(c - \frac{1}{2}\delta_c, c + \frac{1}{2}\delta_c)$ as c varies over the points of the compact set. By Heine–Borel, there are finitely many points c_1, ..., c_n such that the corresponding open intervals cover the compact set. Let δ be the minimum of $\frac{1}{2}\delta_{c_1}$, ..., $\frac{1}{2}\delta_{c_n}$.
Claim: If x and y are any two points of the set, and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Hence f is uniformly continuous. □

Verification of claim

Suppose $|x - y| < \delta$. By construction, there is some point c_j such that $|x - c_j| < \frac{1}{2}\delta_{c_j}$. But $|x - y| < \delta \leq \frac{1}{2}\delta_{c_j}$, so the triangle inequality implies that $|y - c_j| < \delta_{c_j}$.

By the choice of δ_{c_j}, both $|f(y) - f(c_j)| < \frac{1}{2}\varepsilon$ and $|f(x) - f(c_j)| < \frac{1}{2}\varepsilon$.

The triangle inequality implies that $|f(x) - f(y)| < \varepsilon$, as claimed.