Examination 1

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. This problem concerns the ordered field $\mathbb{Q}$, the rational numbers. Your task is to exhibit a concrete example of a bounded subset of $\mathbb{Q}$ that does not have a least upper bound in $\mathbb{Q}$.

2. Suppose that A and B are bounded intervals in $\mathbb{R}$ having non-empty intersection C. Show that $\sup(C)$ equals the minimum of the two numbers $\sup(A)$ and $\sup(B)$.

3. For each of the following scenarios, exhibit an example that satisfies the stated property.
   a) A null sequence of real numbers that is not monotonic.
   b) A monotonic sequence of real numbers that has no convergent subsequence.
   c) An unbounded sequence that has a convergent subsequence.

4. Prove carefully that when (x_n) is a convergent sequence of real numbers, the sequence $(|x_n|)$ of absolute values is convergent too.

5. Suppose $x_n = \dfrac{n^2-1}{n^2+1} + \cos\left(\dfrac{n\pi}{3}\right)$ for each positive integer n. Determine $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

6. State
   a) the Bolzano–Weierstrass theorem, and
   b) Cauchy's criterion for convergence of a sequence of real numbers.

Extra Credit Problem. In this problem, the universe is the power set of $\mathbb{R}$, that is, the set of all subsets of the real numbers. The two operations on sets, $\cup$ and $\cap$ (union and intersection), are somewhat analogous to addition and multiplication. The empty set serves as an identity element for union, since $\emptyset \cup A = A \cup \emptyset = A$ for every set A; the whole set $\mathbb{R}$ serves as an identity element for intersection, since $\mathbb{R} \cap A = A \cap \mathbb{R} = A$ for every set A. The subset relation $\subseteq$ provides an order on sets: a set A is "less than or equal to" a set B if A is a subset of B. The least upper bound of a collection of sets is their union; the greatest lower bound of a collection of sets is their intersection.

Does the power set of $\mathbb{R}$, provided with the operations $\cup$ and $\cap$ and the order $\subseteq$, form a complete ordered field? Explain why or why not.