Examination 1

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. This problem concerns the ordered field $\mathbb{Q}$, the rational numbers. Your task is to exhibit a concrete example of a bounded subset of $\mathbb{Q}$ that does not have a least upper bound in $\mathbb{Q}$.

   Solution. One example is $\{ x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2 \}$. This set is bounded below by 0 and above by 2 (for instance). In $\mathbb{R}$, the least upper bound of the set is $\sqrt{2}$, but $\sqrt{2}$ is an irrational number, so the set has no least upper bound within the universe $\mathbb{Q}$. (We did essentially this example in class on January 19.)

2. Suppose that A and B are bounded intervals in $\mathbb{R}$ having non-empty intersection C. Show that $\sup(C)$ equals the minimum of the two numbers $\sup(A)$ and $\sup(B)$.

   Solution. If $c \in C$, then in particular $c \in A$, so $c \leq \sup(A)$; and similarly $c \leq \sup(B)$. Therefore both $\sup(A)$ and $\sup(B)$ are upper bounds for the set C, so whichever of these two numbers is the smaller one is an upper bound for C.

   What remains to show is that no number M smaller than $\min(\sup(A), \sup(B))$ is an upper bound for C. To address this point, fix an element c_0 of C. (By hypothesis, the set C is not the empty set, so c_0 exists.)

   If $M < c_0$, then M is certainly not an upper bound for C, so there is no loss of generality in supposing that $c_0 \leq M < \min(\sup(A), \sup(B))$. Since $M < \sup(A)$, there is an element a of the set A such that $M < a$. Since the set A is an *interval*, the set A contains the closed interval $[c_0, a]$. Similarly, there is an element b of the set B such that $M < b$, and the interval B contains the interval $[c_0, b]$. Accordingly, the intersection $A \cap B$ contains the interval $[c_0, \min(a, b)]$. In particular, the number $\min(a, b)$ is an element of C that is greater than M, so M is not an upper bound for C.

   What has been shown is that $\min(\sup(A), \sup(B))$ is an upper bound for the set C, and no smaller number is an upper bound for C. The value $\min(\sup(A), \sup(B))$ thus equals $\sup(C)$ by the definition of supremum.

   Remark. Your solution needs to use the assumption that the sets are intervals, for the conclusion is not true for general sets. For example, if A is the doubleton set $\{1, 2\}$, and B is the doubleton set $\{1, 3\}$, then $\sup(A \cap B) = 1$, but $\min(\sup(A), \sup(B)) = 2$.

   You could start by naming the endpoints of the intervals A and B. There is, however, the complication that if the interval A has endpoints a_1 and a_2, then A might be the open interval (a_1, a_2) or the closed interval $[a_1, a_2]$ or one of the intervals $[a_1, a_2)$ and $(a_1, a_2]$; and the same complication arises for B.

   On the other hand, one simplification is possible. Since both the hypothesis and the conclusion are symmetric in A and B, you could start by saying, "There is no loss of generality in supposing that $\sup(A) \leq \sup(B)$."

Examination 1

3. For each of the following scenarios, exhibit an example that satisfies the stated property.

   a) A null sequence of real numbers that is not monotonic.

   Solution. Take x_n equal to $(-1)^n/n$, for example. The sequence (x_n) is not monotonic, for the terms alternate in sign; but the sequence is null because $|x_n| < \varepsilon$ whenever $n > 1/\varepsilon$.

   b) A monotonic sequence of real numbers that has no convergent subsequence.

   Solution. One example is the sequence (n) of natural numbers. The sequence is strictly increasing but unbounded, so every subsequence is unbounded, whence no subsequence can converge.

   c) An unbounded sequence that has a convergent subsequence.

   Solution. Take x_n equal to $(1 + (-1)^n) \cdot n$, for example. The subsequence of terms with n being even is unbounded. The subsequence of terms with n being odd is the constantly 0 sequence, which converges trivially.

4. Prove carefully that when (x_n) is a convergent sequence of real numbers, the sequence $(|x_n|)$ of absolute values is convergent too.

Solution. **Method 1.** If you know the reverse triangle inequality of Theorem 2.9.2(11) on page 30, then you can argue as follows. Fix a positive tolerance ε. The convergent sequence (x_n) is a Cauchy sequence (by Theorem 3.6.1), so there exists an N such that $|x_n - x_m| < \varepsilon$ when $n \geq N$ and $m \geq N$. But

$$\Big||x_n| - |x_m|\Big| \leq |x_n - x_m|,$$

so $\Big||x_n| - |x_m|\Big| < \varepsilon$ when $n \geq N$ and $m \geq N$. Thus the sequence $(|x_n|)$ is a Cauchy sequence of real numbers, hence is convergent (by Theorem 3.6.1 again).

Method 2. An alternative method is to go back to the definition of limit and use that the absolute value is defined by cases. By hypothesis, there is a real number L such that $x_n \to L$. Either $L = 0$, or $L > 0$, or $L < 0$.

If $L = 0$, then (x_n) is a null sequence. Fix a positive ε. The definition of null sequence implies the existence of an N such that $|x_n| < \varepsilon$ when $n \geq N$. This property implies that the sequence $(|x_n|)$ is a null sequence too.

If $L > 0$, then apply the definition of limit with ε equal to the positive number $L/2$. There exists an N such that $|x_n - L| < L/2$ when $n \geq N$, equivalently $0 < L/2 < x_n < 3L/2$ when $n \geq N$. Thus x_n is ultimately positive, so $|x_n| = x_n$ ultimately. Consequently, convergence of (x_n) is the same as convergence of $(|x_n|)$ when $L > 0$.

Examination 1

If $L < 0$, then apply the definition of limit with ε equal to the positive number $-L/2$. There exists an N such that $|x_n - L| < -L/2$ when $n \ge N$, equivalently $3L/2 < x_n < L/2 < 0$ when $n \ge N$. Thus x_n is ultimately negative, so $|x_n| = -x_n$ ultimately. Fix a positive ε. Choose M so large that $|x_n - L| < \varepsilon$ when $n \ge M$. If $n \ge \max(N, M)$, then

$$\big||x_n| - |L|\big| = |-x_n + L| = |x_n - L| < \varepsilon.$$

Accordingly, the definition of limit implies that $|x_n| \to |L|$.

5. Suppose $x_n = \dfrac{n^2-1}{n^2+1} + \cos\left(\dfrac{n\pi}{3}\right)$ for each positive integer n. Determine $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

Solution. The fraction $\dfrac{n^2-1}{n^2+1}$ evidently has limit equal to 1, so

$$\limsup x_n = 1 + \limsup \cos\left(\frac{n\pi}{3}\right) \qquad \text{and} \qquad \liminf x_n = 1 + \liminf \cos\left(\frac{n\pi}{3}\right).$$

The values of the cosine are always between -1 and 1, and each of the extreme values is taken frequently. Therefore $\limsup x_n = 2$, and $\liminf x_n = 0$.

6. State
   a) the Bolzano–Weierstrass theorem, and
   b) Cauchy's criterion for convergence of a sequence of real numbers.

Solution. See Theorem 3.5.9 and Theorem 3.6.1.

Extra Credit Problem. In this problem, the universe is the power set of $\mathbb{R}$, that is, the set of all subsets of the real numbers. The two operations on sets, $\cup$ and $\cap$ (union and intersection), are somewhat analogous to addition and multiplication. The empty set serves as an identity element for union, since $\varnothing \cup A = A \cup \varnothing = A$ for every set A; the whole set $\mathbb{R}$ serves as an identity element for intersection, since $\mathbb{R} \cap A = A \cap \mathbb{R} = A$ for every set A. The subset relation $\subseteq$ provides an order on sets: a set A is "less than or equal to" a set B if A is a subset of B. The least upper bound of a collection of sets is their union; the greatest lower bound of a collection of sets is their intersection.

Does the power set of $\mathbb{R}$, provided with the operations $\cup$ and $\cap$ and the order $\subseteq$, form a complete ordered field? Explain why or why not.

Solution. The indicated structure is not even a field, for inverses are lacking. If A is a non-empty set, then there is no set S for which $A \cup S = \varnothing$; and if A is a proper subset of $\mathbb{R}$, then there is no set S for which $A \cap S = \mathbb{R}$.

Although not a field, this structure is an example of a complete *lattice*, a topic outside the scope of the course.