Final Examination

Instructions. Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. State three definitions or theorems that use the phrase "for every positive ε."

   Solution. Some examples (which you can look up in the textbook) are the definition of limit of a sequence, the definition of null sequence, the definition of Cauchy sequence, the definition of limit of a function, the definition of continuous function, the definition of uniformly continuous function, and the definition of the Riemann integral.

2. For each part, give an example of a non-empty subset S of $\mathbb{R}$ that satisfies the property.

   a) The set S has infinitely many limit points and also has empty interior.

   Solution. One example is $\{ x \in \mathbb{Q} : 0 < x < 1 \}$. The interior is empty, since the set contains no intervals. On the other hand, the limit points are all the points of the closed interval $[0, 1]$, which evidently is an infinite set.

   b) The set S is bounded, and $\sup\{ x^2 : x \in S \} \neq (\sup S)^2$.
   (As usual, the notation "sup" means the supremum, that is, the least upper bound.)

   Solution. One example is the interval $(-1, 0)$. The supremum is 0, so the square of the supremum is 0. On the other hand, the supremum of the set of squares is 1.

3. Consider the sequence defined recursively as follows:

   $$x_1 = \sin(1), \qquad \text{and} \qquad x_{n+1} = \sin(x_n) \quad \text{when } n \geq 1.$$

   Does this sequence of real numbers converge? Explain why or why not.
   (You may assume the standard properties of the sine function shown on the second page.)

   Solution. The sine function maps the interval $(0, \pi/2)$ into the interval $(0, 1)$, a subset of $(0, \pi/2)$, so subsequent applications of the sine function continue to have image contained in $(0, 1)$. Thus all terms of the given sequence lie between 0 and 1.

   Now $\sin(x) < x$ when $0 < x \leq 1$, as follows from the picture. You can prove this inequality by invoking the mean-value theorem: since the derivative of the sine function is the cosine,

   $$\frac{\sin(x) - \sin(0)}{x - 0} = \cos(c) \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

   But $\cos(c) < 1$ when $0 < c < 1$, so multiplying by the positive number x implies that $\sin(x) < x$.

   Accordingly, $x_{n+1} = \sin(x_n) < x_n$ for each n, so the given sequence is decreasing. Being a bounded monotonic sequence, the sequence necessarily converges.

   Moreover, the limit L is a nonnegative number that is a fixed point of the sine function: $L = \sin(L)$. Since $\sin(x)$ is strictly less than x when x is strictly positive, the value of the limit L must be 0.

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4. Here are five concepts from this course commencing with the consonant c:
   a) closure of a set
   b) compact set
   c) completeness axiom
   d) continuous function
   e) covering of a set

   Explain the meaning of **three** of these concepts.

   Solution. The definitions can be located from the index of the textbook.

5. When E is a non-empty subset of $\mathbb{R}$, the distance-to-E function d_E is defined as follows:

$$d_E(x) = \inf\{\, |x-y| : y \in E \,\} \qquad \text{for each real number } x.$$

Prove that if the set E is closed, then "inf" can be replaced by "min" in this definition: in other words, for each x, the infimum (greatest lower bound) is attained for some y in E.

Solution. Method 1. By the definition of infimum, there is a sequence (y_n) in E such that $|x-y_n| \to d_E(x)$. This sequence is necessarily bounded: namely, $|x-y_n| \le d_E(x)+1$ for all but finitely many values of n, so the triangle inequality implies that $0 \le |y_n| \le |x|+d_E(x)+1$ for all but finitely many values of n.

By the Bolzano–Weierstrass theorem, there is a convergent subsequence (y_{n_k}). Since the set E is closed, the limit y^* of this convergent subsequence lies in E. Every subsequence of the convergent sequence $(|x-y_n|)$ converges to the same limit as the whole sequence, so $|x-y_{n_k}| \to d_E(x)$.

The absolute-value function is continuous, and continuous functions preserve convergent sequences, so

$$d_E(x) = \lim_{k\to\infty} |x-y_{n_k}| = |x-y^*|.$$

Thus the minimal distance $d_E(x)$ is attained at the point y^* in E.

Remark. The absolute-value function not only is continuous but actually is uniformly continuous: by the triangle inequality, $\big||t_1|-|t_2|\big| \le |t_1-t_2|$ for arbitrary points t_1 and t_2, so δ can be taken equal to ε in the definition of uniform continuity.

Method 2. Observe that if a non-empty *closed* set is bounded above, then the supremum of the set must lie in the set. (Proof: there is a sequence of points of the set converging to the supremum, and the limit lies in the set since the set is closed.) Similarly, if a non-empty closed set is bounded below, then the infimum of the set lies in the set.

Now $E \cap [x,\infty)$ is closed, being the intersection of two closed sets. And this intersection is bounded below (by x). Therefore $E \cap [x,\infty)$, if non-empty, contains a minimal element, say y_1. Evidently $|x - y_1|$, the distance between x and y_1, is the infimum (also minimum) of the values $|x - y|$ as y runs over the set $E \cap [x, \infty)$.

Similarly, $E \cap (-\infty, x]$ is a closed set that is bounded above, so this intersection (if nonempty) contains a maximal element, say y_2. And $|x - y_2|$, the distance between x and y_2, is the infimum (also minimum) of the values $|x - y|$ as y runs over the set $E \cap (-\infty, x]$.

Accordingly, the minimal distance $d_E(x)$ is the minimum of $|x - y_1|$ and $|x - y_2|$, and this value is attained at one of the points y_1 and y_2 in E.

6. When $f : [0, 2] \to \mathbb{R}$ is a monotonic, differentiable function for which $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$, what (if anything) can be deduced about
   a) the limit $\lim_{x\to 1} f(x)$?
   b) the derivative $f'(1)$?
   c) the integral $\int_0^2 f(x)\,dx$?

   Explain your reasoning.

Solution.

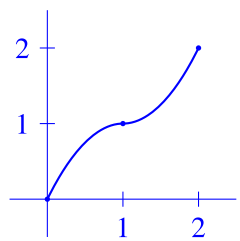
a) A differentiable function is necessarily continuous, and continuous functions preserve limits, so $\lim_{x\to 1} f(x) = f(1) = 1$.

b) Since f is monotonic, and $f(0) < f(1)$, the function f must be (weakly) increasing. Therefore $f'(1)$, being the limit

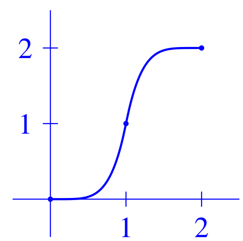
$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

of nonnegative fractions, must be a nonnegative number.

Nothing more can be said without more information about f. Indeed, the pictures below indicate that $f'(1)$ can be an arbitrary nonnegative number for a suitable choice of f.



$f'(1) = 0$



$f'(1)$ is large

c) A differentiable function is continuous, hence integrable, so $\int_0^2 f(x)\,dx$ is at least a well-defined quantity. Some bounds on the integral can be obtained by approximating the area under the graph by areas of rectangles.

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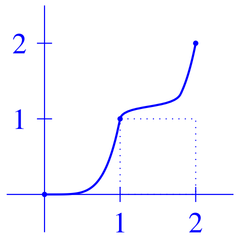
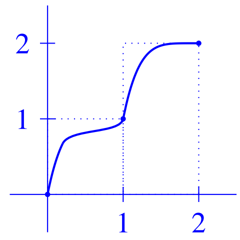
Since f is (weakly) increasing, $f(x) \le 1$ when $x \le 1$ and $f(x) \le 2$ when $1 < x \le 2$. Therefore the area under the curve is at most 3. On the other hand, $f(x) \ge 1$ when $x \ge 1$, so the area under the curve is at least 1.

Thus $1 \le \int_0^2 f(x)\,dx \le 3$. Actually, the extreme values cannot be achieved, for $f(x)$ is continuous and has a value strictly less than 1 when $x = 0$ and a value strictly greater than 1 when $x = 2$. The pictures below indicate that every value strictly between 1 and 3 can be obtained as a value of the integral for a suitable choice of f.



area is a little more than 1



area is a little less than 3

Extra Credit Problem. Suppose $f : (-1, 1) \to \mathbb{R}$ is defined as follows: $f(x) = \dfrac{x^{409}+1}{x^{501}+1}$. Say as much as you can about the image, the set $\{\, f(x) : -1 < x < 1 \,\}$.

Solution. Since the denominator equals 0 only when $x = -1$ (assuming that x is a real number), and the point -1 is outside the domain of f, the function f is continuous on the indicated domain. The intermediate-value theorem implies that the image of f is an interval. Which interval?

Observe that $f(0) = 1 = f(1)$. Moreover, if $0 < x < 1$, then $x^{501} < x^{409}$, whence $f(x) > 1$. The continuous function f on the compact interval $[0, 1]$ therefore attains an absolute maximum at some interior point of this interval, and this maximum value is strictly greater than 1.

Since f is a differentiable function on the interval $[0, 1]$, the maximum occurs at an interior point of this interval where the derivative f' is equal to 0. The location of the critical point can be determined in principle by studying the numerator of $f'(x)$, which equals

$$(x^{501}+1)\cdot 409x^{408} - (x^{409}+1)\cdot 501x^{500}, \qquad \text{or}$$
$$x^{408}(-92x^{501} - 501x^{92} + 409).$$

When $x > 0$, the derivative $f'(x)$ equals 0 if and only if $92x^{501} + 501x^{92} = 409$. The left-hand side of this equation is continuous and strictly increasing on the interval $[0, 1]$, with a value of 0 when $x = 0$ and a value of 593 when $x = 1$, so the equation has a unique solution b between 0 and 1. But there is no hope of computing the value of b exactly; numerical calculation shows that $b \approx 0.9972$.

What is the behavior of the function f on the interval $(-1, 0)$? On this interval, $|x^{501}| < |x^{409}| < 1$, but the values of x^{501} and x^{409} are negative, so $0 < f(x) < 1$. Moreover, l'Hôpital's rule implies that

$$\lim_{x\to -1} f(x) = \frac{409}{501}.$$

The sign of the derivative $f'(x)$ is the same as the sign of the expression $-92x^{501} - 501x^{92} + 409$. This expression itself has derivative equal to $92 \cdot 501 \cdot (-x^{500} - x^{91})$, which is positive when $-1 < x < 0$, so the expression $-92x^{501} - 501x^{92} + 409$ is increasing on the interval $(-1, 0)$; since this expression takes the value 0 when $x = -1$, the expression is positive on the interval $(-1, 0)$. Accordingly, f' is positive on the interval $(-1, 0)$, so the original function f is increasing on this interval.

Putting all the above deductions together shows that the image of the interval $(-1, 1)$ is the interval

$$\left(\frac{409}{501}, \frac{b^{409} + 1}{b^{501} + 1}\right],$$

where b is the unique positive value of x for which $92x^{501} + 501x^{92} = 409$.

Remark. If you try to plot a graph of $\dfrac{x^{409} + 1}{x^{501} + 1}$ using a computer, you will get a picture that looks like a flat line at height 1, perhaps with some weirdness near the endpoints -1 and 1. As far as the computer is concerned, the high powers x^{409} and x^{501} are effectively equal to 0 on the interval in question. But you now know better: the function takes some values a little less than 1 as well as some values a little bigger than 1.

Background for Problem 3. You may assume as standard knowledge both the graph below and the fact that $\frac{d}{dx}\sin(x) = \cos(x) = \sin\left(\frac{1}{2}\pi - x\right)$.