Examination 2

Part A: Sentence Completion

Your answer to each of problems 1–3 should be a complete sentence that starts as indicated.

1. The Bolzano–Weierstrass theorem states that every ....
2. To say that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence means that for every positive ε ....
3. The statement "$\lim_{x\to c} f(x) = L$" means that c is a cluster point of the domain of f and ....

Part B: Examples

Your task in problems 4–5 is to exhibit a concrete example satisfying the indicated property. You should provide a brief explanation of why your example works.

4. Give an example of a bounded sequence $\{x_n\}_{n=1}^{\infty}$ having the property that
$$\sup\{x_n : n \geq 1\} \neq \limsup_{n\to\infty} x_n.$$
5. Give an example of a sequence $\{x_n\}_{n=1}^{\infty}$ having the properties that $x_n > 0$ for every natural number n, and the series $\sum_{n=1}^{\infty} x_n$ converges, and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. (In other words, the ratio test fails to prove convergence of the series, but the series does converge nonetheless.)

Part Γ: Proof

6. Find a positive number δ having the property that $\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{9}$ whenever $|x-2| < \delta$. Explain why your δ works.

Part Δ: Optional Extra Credit Problem

The capital Greek letter Σ (Sigma) traditionally denotes a Sum, and the capital Greek letter Π (Pi) similarly denotes a Product. A plausible meaning to attach to the notation $\prod_{n=1}^{\infty} a_n$ is $\lim_{N\to\infty} \prod_{n=1}^{N} a_n$, that is, the limit of the sequence of partial products. If this limit exists, then the infinite product can be said to converge.

Does the infinite product $\prod_{n=1}^{\infty} \left(1 + \frac{1}{2^n}\right)$ converge? Explain why or why not.