Reminder

The first exam takes place in class on February 16
(a week from Friday).

A first theorem about existence of limits

Theorem (Monotone Convergence)

If a sequence is increasing and bounded above, then the sequence converges (and the limit is the least upper bound of the terms).

Example

Prove that if $x_1 = 1$, and $x_{n+1} = \sqrt{2 + x_n}$ when $n \geq 1$, then $\lim_{n\to\infty} x_n$ exists.

Sketch of proof by induction.

Basis step for increasing: $x_2 = \sqrt{2+1} > 1 = x_1$.
Induction step for increasing: Suppose n is a natural number for which $x_{n+1} \geq x_n$. Then $x_{n+2} = \sqrt{2 + x_{n+1}} \geq \sqrt{2 + x_n} = x_{n+1}$.
Basis step for bounded above by 4: $x_1 = 1 < 4$.
Induction step for bounded above by 4: If n is a natural number for which $x_n \leq 4$, then $x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2+4} < 4$. $\square$

Continuation

Theorem

If a sequence is decreasing and bounded below, then the sequence converges (and the limit is the greatest lower bound of the terms).

In the preceding example, if x_1 is changed to be 3, then the recursively defined sequence will be decreasing, with limit 2.

Assignment due next class

1. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence with the property that every subsequence converges to some limit, possibly different limits for different subsequences. Prove that in fact all the subsequences must have the same limit. [not hard]
2. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence, and L is a number with the property that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ has a subsubsequence $\{x_{n_{k_i}}\}_{i=1}^{\infty}$ that converges to L. Prove that the original sequence $\{x_n\}_{n=1}^{\infty}$ must be convergent. [hard]
3. Read subsection 2.2.3 in the textbook.