Cauchy sequences

A sequence $\{x_n\}_{n=1}^{\infty}$ is a *Cauchy sequence* if the terms in the tail eventually get arbitrarily close together: namely, $\forall \varepsilon > 0 \;\; \exists M$ such that $|x_m - x_n| < \varepsilon$ when $n \geq M$ and $m \geq M$.

Remark

The inequality $|x_n - x_{n+1}| < \varepsilon$ is insufficient.
Example: $x_n = \log(n)$. Then $|x_{n+1} - x_n| = \log\left(\frac{n+1}{n}\right)$, which gets close to 0 when n gets large, but $|x_{2n} - x_n| = \log\left(\frac{2n}{n}\right) = \log(2)$, which is not close to 0.

More on Cauchy sequences

A sequence of *real numbers* converges if and only if it is a Cauchy sequence.

Why? Suppose we have a Cauchy sequence. Fix a positive ε. Find M such that $|x_m - x_n| < \varepsilon/2$ when $n \geq M$ and $m \geq M$.

The terms in the M-tail of the sequence are bounded between $x_M - \frac{\varepsilon}{2}$ and $x_M + \frac{\varepsilon}{2}$, and the other terms are bounded between $\pm \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|\}$. Thus the sequence is bounded, so there is a monotonic subsequence that converges to some limit L.

There is some value of n greater than M for which $|x_n - L| < \frac{\varepsilon}{2}$. So if $m \geq M$, then

$$|x_m - L| = |(x_m - x_n) + (x_n - L)| \leq |x_m - x_n| + |x_n - L| < \varepsilon.$$

So the original sequence fits the definition of convergence.

Assignment due next class

- Write solutions to Exercises 2.4.1 and 2.5.1.