Sequences and series

A *sequence* means a list of numbers: x_1, x_2, x_3, ...
Notation: $\{x_n\}_{n=1}^{\infty}$

A *series* means a sum of a list: $x_1 + x_2 + x_3 + \cdots$.
Notation: $\sum_{n=1}^{\infty} x_n$

A *series* converges when the *sequence of partial sums* converges:

$$x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3,\ \ldots.$$

Geometric series

You showed that $\sum_{k=0}^{n-1} r^k = \frac{1 - r^n}{1 - r}$ when $r \neq 1$ [Exercise 2.5.1].

Taking the limit with respect to n shows that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \qquad \text{when} \qquad |r| < 1$$

Example

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = 3.$$

Cauchy's condensation test

If $\{x_n\}_{n=1}^{\infty}$ is a **decreasing** *sequence* of positive numbers, then the *series* $\sum_{n=1}^{\infty} x_n$ converges if and only if the series $\sum_{n=1}^{\infty} 2^n x_{2^n}$ converges.

Example

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

Since $\frac{1}{n^2}$ decreases when n increases, the question reduces to convergence of the series $\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^2}$. This new series equals $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent geometric series. So the original series converges too.

Note. The limits are different: $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, but $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Assignment due next class

- Read subsection 2.5.1 in the textbook.
- Write solutions to Exercises 2.4.7 and 2.5.3(a).