Thinking positively

If $x_k \geq 0$ for every k, then the partial sums $\sum_{k=1}^{n} x_k$ increase when n increases, so the corresponding series $\sum_{k=1}^{\infty} x_k$ converges if and only if the partial sums are bounded above.

Example

$\sum_{n=1}^{\infty} \frac{(\cos n)^2}{2^n}$ converges because $0 < (\cos n)^2 < 1$, so the partial sums of the series are less than the partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (a geometric series), which are bounded above by 1.

Non-example

$\sum_{n=1}^{\infty} (-1)^n$ is a divergent series that has bounded partial sums.

Cauchy's condensation test (not in the book)

If $\{x_n\}_{n=1}^{\infty}$ is a **decreasing** sequence of positive numbers, then the *series* $\sum_{n=1}^{\infty} x_n$ converges if and only if the series $\sum_{n=1}^{\infty} 2^n x_{2^n}$ converges.

Why? Monotonicity implies that

$$\frac{8x_8}{2} = 4x_8 \leq x_4 + x_5 + x_6 + x_7 \leq 4x_4$$

and in general

$$\frac{2^{n+1}x_{2^{n+1}}}{2} \leq x_{2^n} + \cdots + x_{2^{n+1}-1} \leq 2^n x_{2^n}.$$

The double inequality shows that the partial sums of the original series are bounded above if and only if the partial sums of the condensed series are bounded above.

Assignment due next class

- Read the rest of section 2.5 in the textbook.