Some standard convergence tests for positive series

- geometric series
- comparison test
- Cauchy's condensation test for monotonic series [not in book]
- root test
- ratio test

Useful example: *p*-series

When p is a constant, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

- converges if $p > 1$,
- diverges if $p \leq 1$.

The proof in the book is Cauchy's condensation test in disguise.

Remark

When $p > 1$, the value of the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is known as $\zeta(p)$, the so-called Riemann zeta function.

Roger Apéry (1916–1994) became famous by proving in 1978 that $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$, or $\zeta(3)$, is an irrational number.

Cauchy's root test

Suppose $x_n \geq 0$ for every n. Then the series $\sum_{n=1}^{\infty} x_n$

- converges if $\limsup_{n\to\infty} x_n^{1/n} < 1$
- diverges if $\limsup_{n\to\infty} x_n^{1/n} > 1$.

Example

$\sum_{n=1}^{\infty} \left(1 - (-1)^{\omega(n)}\right) \frac{n}{2^n}$, where $\omega(n)$ denotes the number of distinct prime factors of n. [For $\omega(n)$, see `http://oeis.org/A001221`.]

The expression $\left(1 - (-1)^{\omega(n)}\right)$ is infinitely often 0 and infinitely often 2. The lim sup equals $\lim_{n\to\infty} \left(\frac{2n}{2^n}\right)^{1/n} = \frac{1}{2} < 1$. The series converges.

Assignment due next class

1. When is the second exam?
2. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of strictly positive real numbers. Define numbers α, β, γ, and δ as follows:

$$\alpha = \liminf_{n\to\infty} \frac{x_{n+1}}{x_n}, \quad \beta = \liminf_{n\to\infty} x_n^{1/n},$$
$$\gamma = \limsup_{n\to\infty} x_n^{1/n}, \quad \delta = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

   Prove that $\alpha \leq \beta \leq \gamma \leq \delta$.

Hint: To show that $\beta \leq \gamma$ is easy. If you can prove one of the remaining two inequalities, then you can prove the other one by symmetry. So the main issue is to show that $\gamma \leq \delta$. It suffices to show for an arbitrary positive ε that $\gamma \leq \delta + \varepsilon$.