True or false?

1. If $f : [a, b] \to \mathbb{R}$ is continuous, then the range of f is some interval $[c, d]$.
   True by combining intermediate-value theorem with extreme-value theorem (special case: for a constant function, the interval $[c, d]$ degenerates to a point).

2. A continuous function maps convergent sequences to convergent sequences.
   [assuming the limit of the sequence belongs to the domain of the function]
   True: equivalent formulation of the notion of continuity.

3. A continuous function maps Cauchy sequences to Cauchy sequences.
   Depends on the domain of f.
   If domain of f is all of $\mathbb{R}$, then true.

Counterexample to 3

Consider $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ and the sequence $\{1/n\}_{n=1}^{\infty}$.
This is a Cauchy sequence in the domain.
But the image sequence $\{f(1/n)\}_{n=1}^{\infty}$ is not a Cauchy sequence.

The quantified formulation of continuity

f is continuous at c if for every positive ε there exists a positive δ such that the inequality $|x - c| < \delta$ implies the inequality $|f(x) - f(c)| < \varepsilon$.

The value of δ is allowed to depend on ε and on c and on f but not on x.

If δ can be chosen to be independent of the point c in the domain of f, then f is called *uniformly continuous*.

A uniformly continuous function does map Cauchy sequences to Cauchy sequences.

A third important property of a continuous function whose domain is a closed, bounded interval

Theorem

Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous. Then

1. *f attains a maximum value and attains a minimum value (extreme-value theorem);*
2. *the range of f is an interval or a single point (intermediate-value theorem); and*
3. *f is automatically uniformly continuous.*

Assignment due next class

Write a solution to Exercise 3.4.10.