Recap: Three views of the derivative

Suppose $f\colon I \to \mathbb{R}$ (where I is an interval), and c is an interior point of I. Then f is *differentiable at c* when any of the following equivalent properties holds.

1. The limit $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists, in which case the limit is the *derivative*, usually denoted by $f'(c)$.
2. There exists a function $F_c\colon I \to \mathbb{R}$ with the properties that F_c is continuous at c and $f(x) = F_c(x)(x - c) + f(c)$ for every x. In this situation, $f'(c) = F_c(c)$.
3. There exists a "best linear approximation" of f at c: namely, a linear function T_c with the property that

   $$\lim_{h \to 0} \frac{f(c + h) - f(c) - T_c(h)}{h} = 0.$$

   Here $T_c(h) = f'(c)h$ (multiplication by the constant $f'(c)$).

Application: the product rule

If f and g are two functions differentiable at c, then property 2 provides functions F and G (continuous at c) such that

$$f(x) = F(x)(x-c) + f(c)$$
$$g(x) = G(x)(x-c) + g(c).$$

Multiply: $f(x)g(x) =$
$[F(x)G(x)(x-c) + F(x)g(c) + f(c)G(x)](x-c) + f(c)g(c)$.
The expression in brackets is continuous at c, so the product function fg is differentiable at c, and the derivative equals $0 + F(c)g(c) + f(c)G(c)$, or $\boxed{f'(c)g(c) + f(c)g'(c)}$.

Exercise

Are the following functions differentiable at the point 0?

1. $f(x) = x|x|$
2. $g(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$
3. $h(x) = \begin{cases} x^2\cos(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

Assignment due next class

Use definition 2 of derivatives to prove the quotient rule.