Recap: Cauchy's version of the mean-value theorem

If f and g are two continuous functions on $[a, b]$ that are differentiable at all points of (a, b), then there is a point c in (a, b) for which

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

If $g(x)$ is the identity function x, then the conclusion reduces to

$$f(b) - f(a) = f'(c)(b - a),$$

the basic version of the mean-value theorem.

l'Hôpital's rule

Suppose f and g are differentiable functions on an interval, except perhaps at one point b.

Suppose additionally that $\lim_{x \to b} f(x) = 0 = \lim_{x \to b} g(x)$ and that both $g(x)$ and $g'(x)$ are different from 0 when x is in some punctured neighborhood of b.

If $\lim_{x \to b} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to b} \frac{f(x)}{g(x)}$ does too, and these limits agree.

Examples of l'Hôpital's rule

- $\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2\cos(2x)}{1} = 2$
- $\lim_{x \to 0} \frac{\cos(x) - 1 + \frac{1}{2}x^2}{x^4} = \frac{1}{24}$ by multiple applications of l'Hôpital's rule

Proof of l'Hôpital's rule

The functions f and g have "removable discontinuities" at b: defining $f(b)$ and $g(b)$ to be 0 makes f and g continuous at b.
For each x, Cauchy's form of the mean-value theorem applies on the interval between b and x: there is a point c_x between b and x such that $g'(c_x)(f(x) - f(b)) = f'(c_x)(g(x) - g(b))$, or $g'(c_x)f(x) = f'(c_x)g(x)$.
Then $\dfrac{f(x)}{g(x)} = \dfrac{f'(c_x)}{g'(c_x)}$ (division being possible by the hypothesis that $g(x)$ and $g'(c_x)$ are not equal to 0).
When $x \to b$, so does the in-between point c_x.
So the existence of the limit of the ratio of derivatives implies the existence of $\lim\limits_{x\to b} \dfrac{f(x)}{g(x)}$, and the two limits match.

Introduction to Taylor's theorem

Replace b with x in the statement of the mean-value theorem:

$$\frac{f(x) - f(a)}{x - a} = f'(c),$$

or, equivalently,

$$f(x) = f(a) + f'(c)(x - a)$$

(where c depends on x).

Second-order generalization: If f'' (the second derivative) exists on some interval containing a and x, then there is some c between a and x for which

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(c)(x - a)^2.$$

Assignment due next class

Review exercises (not to hand in):

1. Why does the *sequence* $\left\{\dfrac{2^n}{n!}\right\}_{n=1}^{\infty}$ converge?
2. Why does the *series* $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!}$ converge?
3. Determine $\sup\left\{\dfrac{2^n}{n!} : n \in \mathbb{N}\right\}$.
4. Determine $\displaystyle\limsup_{n\to\infty}\frac{2^n}{n!}$.