Darboux's modification of Riemann's amplification of Cauchy's idea for defining the integral

Suppose f is a bounded function on $[a, b]$.

For an arbitrary natural number n, partition the interval into n pieces: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

For each k between 1 and n, let M_k denote $\sup\{f(x) : x_{k-1} \le x \le x_k\}$.

Then $\displaystyle\sum_{k=1}^{n} M_k(x_k - x_{k-1})$ is an *upper sum* for f on $[a, b]$.

Similarly define lower sums by replacing sup with inf.

If the infimum of all the upper sums equals the supremum of all the lower sums, then f is declared to be an *integrable function*, and the indicated value is the integral $\int_a^b f(x)\,dx$ (or $\int_a^b f$ for short).

Practice/review exercises (not to hand in)

(1) Prove by induction that $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

(2) For the function x^2 on the interval $[0,1]$, consider a partition with n subintervals of equal width. Show that the upper sum equals $\displaystyle\sum_{j=1}^{n} \frac{1}{n} \cdot \left(\frac{j}{n}\right)^2$, and the lower sum equals $\displaystyle\sum_{j=0}^{n-1} \frac{1}{n} \cdot \left(\frac{j}{n}\right)^2$.

(3) Deduce from (1) and (2) that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$.