Examination 2

Part A: Sentence Completion

Your answer to each of problems 1–3 should be a complete sentence that starts as indicated.

1. The Bolzano–Weierstrass theorem states that every ....

   Solution. The Bolzano–Weierstrass theorem states that every bounded sequence has a convergent subsequence.

2. To say that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence means that for every positive ε ....

   Solution. To say that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence means that for every positive ε there exists a number M such that $|x_n - x_k| < \varepsilon$ whenever $n \geq M$ and $k \geq M$.

3. The statement "$\lim_{x\to c} f(x) = L$" means that c is a cluster point of the domain of f and ....

   Solution. The statement "$\lim_{x\to c} f(x) = L$" means that c is a cluster point of the domain of f and for every positive ε there exists a positive δ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$ and x lies in the domain of f.

   Equivalently, the statement "$\lim_{x\to c} f(x) = L$" means that c is a cluster point of the domain of f and for every sequence $\{x_n\}_{n=1}^{\infty}$ in the domain of f such that $x_n \neq c$ for every n and $\lim_{n\to\infty} x_n = c$, the image sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Part B: Examples

Your task in problems 4–5 is to exhibit a concrete example satisfying the indicated property. You should provide a brief explanation of why your example works.

4. Give an example of a bounded sequence $\{x_n\}_{n=1}^{\infty}$ having the property that

$$\sup\{x_n : n \geq 1\} \neq \limsup_{n\to\infty} x_n.$$

   Solution. If $x_n = 1/n$, then $\lim_{n\to\infty} x_n = 0$, hence $\limsup_{n\to\infty} x_n = 0$. But $\sup\{x_n : n \geq 1\} = 1$.

   Here is an even more extreme example:

$$x_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

   Again $0 = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, but $\sup\{x_n : n \geq 1\} = 1$.

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5. Give an example of a sequence $\{x_n\}_{n=1}^{\infty}$ having the properties that $x_n > 0$ for every natural number n, and the series $\sum_{n=1}^{\infty} x_n$ converges, and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. (In other words, the ratio test fails to prove convergence of the series, but the series does converge nonetheless.)

Solution. If $x_n = \frac{1}{n^2}$, then the series $\sum_{n=1}^{\infty} x_n$ converges (being a p-series for which $p > 1$), and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1.$$

There are other examples, but this one is the most popular.

Part Γ: Proof

6. Find a positive number δ having the property that $\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{9}$ whenever $|x-2| < \delta$. Explain why your δ works.

Solution. Remark. To show that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$ requires finding, for each positive ε, a corresponding positive δ having the property that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. This problem, however, asks merely for a δ corresponding to one particular value of ε, namely ε equal to $\frac{1}{9}$.

There is more than one correct answer. Indeed, if a certain value of δ works, then so does any smaller positive number. Notice too that x cannot be allowed to become 0 (because then $1/x$ is undefined), so δ certainly should be less than 2.

Method 1: two-step. If δ is chosen to be a number smaller than 1, then the restriction that $|x-2| < \delta$ implies, in particular, that $x > 1$. Therefore

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| \le \left|\frac{2-x}{2}\right|.$$

The upper bound on the right-hand side will be less than $1/9$ if $|x-2| < 2/9$. Since $2/9$ is less than 1, the value $2/9$ is a valid choice for δ.

Method 2: exact solution. If $x \ge 2$, then $1/x \le 1/2$, so

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{1}{2} - \frac{1}{x}.$$

Now

$$\frac{1}{2} - \frac{1}{x} < \frac{1}{9} \iff \frac{7}{18} < \frac{1}{x} \iff x < \frac{18}{7} \iff x - 2 < \frac{4}{7},$$

so when $x \geq 2$, the needed restriction is that $|x-2| < 4/7$. On the other hand, if $0 < x < 2$, then $1/x > 1/2$, so

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{1}{x} - \frac{1}{2}.$$

Now

$$\frac{1}{x} - \frac{1}{2} < \frac{1}{9} \iff \frac{1}{x} < \frac{11}{18} \iff -x < -\frac{18}{11} \iff 2 - x < \frac{4}{11},$$

so when $x < 2$, the needed restriction is that $|x - 2| < 4/11$. Accordingly, if δ is taken equal to $4/11$ (the minimum of $4/7$ and $4/11$), then

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{9} \qquad \text{when} \qquad |x - 2| < \delta,$$

whether $x \geq 2$ or $x < 2$. And $4/11$ is the largest possible value of δ with this property.

Remark. You could shorten Method 2 by observing that the slope of the graph of the function $1/x$ has decreasing magnitude when x increases. Therefore the worst case occurs when $x < 2$, so it was not really necessary to compute the first case in Method 2.

Method 3: guess and check. Since the slope of the graph has small magnitude near the point where $x = 2$, a reasonable guess is that taking δ equal to $1/9$ should work. Instead of checking the inequality for every point x in the interval $(2 - \frac{1}{9}, 2 + \frac{1}{9})$, it suffices to check the endpoints, since the function $1/x$ is monotonic. Now

$$\left|\frac{1}{2 + \frac{1}{9}} - \frac{1}{2}\right| = \left|\frac{9}{19} - \frac{1}{2}\right| = \frac{1}{38} < \frac{1}{9},$$

and

$$\left|\frac{1}{2 - \frac{1}{9}} - \frac{1}{2}\right| = \left|\frac{9}{17} - \frac{1}{2}\right| = \frac{1}{34} < \frac{1}{9}.$$

Thus taking δ equal to $1/9$ does indeed work.

Part Δ: Optional Extra Credit Problem

The capital Greek letter Σ (Sigma) traditionally denotes a Sum, and the capital Greek letter Π (Pi) similarly denotes a Product. A plausible meaning to attach to the notation $\prod_{n=1}^{\infty} a_n$ is $\lim_{N\to\infty} \prod_{n=1}^{N} a_n$, that is, the limit of the sequence of partial products. If this limit exists, then the infinite product can be said to converge.

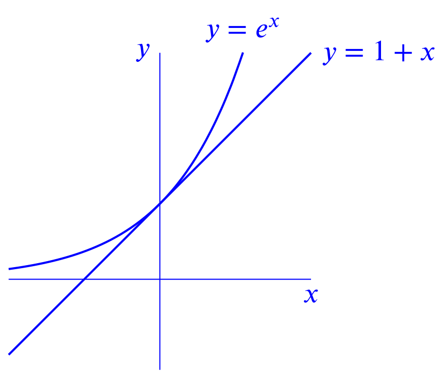
Does the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{2^n}\right)$ converge? Explain why or why not.

Solution. Remark. The standard definition of convergence of an infinite product is actually a bit more involved that what is stated in the problem, because 0 is a special number with respect to multiplication. If a_1 were equal to 0, for example, then every partial product would be equal to 0, whatever the values of a_2, a_3, …. But the notion of convergence ought to depend only on the behavior of a_n when n is large. Whether or not $\prod_{n=1}^{\infty} a_n$ converges ought not to depend on the value of a_1 or even on the values of finitely many terms.

The usual definition of convergence of an infinite product therefore requires that first of all, at most a finite number of terms can be equal to 0; secondly, when these terms are deleted, the partial products of the remaining terms have a limit; and thirdly, this limit is not equal to 0. The reason for the third condition is to maintain the natural property that a product is equal to 0 if and only if one of the terms is equal to 0.

In the problem at hand, each term in the product is larger than 1, so the preceding subtleties do not arise. Moreover, since each term is larger than 1, the partial products form a strictly increasing sequence. Since a bounded monotonic sequence converges, all that needs to be checked to prove convergence of this infinite product is that the partial products remain bounded above. Here are three ways to verify that boundedness.

Method 1. The graph below from elementary calculus shows that $1 + x \le e^x$ for every real number x. Moreover, strict inequality holds when $x \ne 0$. The key features of the graph are that the slope of the exponential function at the origin is equal to 1, and the exponential graph is convex ("concave up"). Therefore the graph of e^x lies above the tangent line passing through the point $(0, 1)$.



The deduction is that $1 + \frac{1}{2^n} < e^{1/2^n}$ for each natural number n. Consequently,

$$\prod_{n=1}^{N}\left(1+\frac{1}{2^n}\right) < \prod_{n=1}^{N} e^{1/2^n} = \exp\left(\sum_{n=1}^{N}\frac{1}{2^n}\right).$$

Now the right-hand side is bounded above by e^1, since the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to the sum 1. Thus the partial products are indeed bounded above, as required.

Method 2. The geometric average of a collection of positive numbers is less than or equal to the arithmetic average, that is,

$$\left(\prod_{n=1}^{N} b_n\right)^{1/N} \leq \frac{1}{N}\sum_{n=1}^{N} b_n.$$

The reason is that when a collection of numbers has a fixed sum, the product of the numbers is maximized when the numbers are all equal to each other (in which case the geometric mean and the arithmetic mean agree).

Apply this principle when $b_n = 1 + \frac{1}{2^n}$ to deduce that

$$\left[\prod_{n=1}^{N}\left(1+\frac{1}{2^n}\right)\right]^{1/N} \leq \frac{1}{N}\sum_{n=1}^{N}\left(1+\frac{1}{2^n}\right) = 1 + \frac{1}{N}\sum_{n=1}^{N}\frac{1}{2^n} < 1 + \frac{1}{N}.$$

Therefore

$$\prod_{n=1}^{N}\left(1+\frac{1}{2^n}\right) < \left(1+\frac{1}{N}\right)^N.$$

The expression on the right-hand side increases to the limit e when N increases, so e is an upper bound for the partial products.

Method 3. This method is longer than the two preceding ones but has the advantage of using only elementary tools.

Convergence is unaffected by the first few terms in the product, so demonstrating boundedness of the partial product $\prod_{n=4}^{N}\left(1+\frac{1}{2^n}\right)$ will do. The reason for discarding the initial terms of the product is that $\frac{1}{2^n} \leq \frac{1}{n^2}$ when $n \geq 4$.

(Proof by induction: The basis step is the observation that $\frac{1}{2^4} = \frac{1}{4^2}$. For the induction step, suppose n is a natural number (at least 4) for which $\frac{1}{2^n} \leq \frac{1}{n^2}$. Then $\frac{1}{2^{n+1}} \leq \frac{1}{2n^2}$. What remains to show is that $\frac{1}{2n^2} \leq \frac{1}{(n+1)^2}$, equivalently that $(n+1)^2 \leq 2n^2$, or that $2n+1 \leq n^2$. But $n \geq 4$, so $n^2 \geq 4n = 2n + 2n > 2n + 1$, as required.)

Accordingly, proving an upper bound for the partial product $\prod_{n=4}^{N}\left(1+\frac{1}{n^2}\right)$ will suffice. First consider a different but related problem of convergence of $\prod_{n=4}^{\infty}\left(1-\frac{1}{n^2}\right)$. (Why this new problem is relevant will become apparent soon.) Observe that

$$1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2} = \frac{(n-1)(n+1)}{n^2}.$$

Therefore

$$\prod_{n=4}^{N} \left(1 - \frac{1}{n^2}\right) = \frac{3 \cdot 5}{4^2} \cdot \frac{4 \cdot 6}{5^2} \cdot \frac{5 \cdot 7}{6^2} \cdots \frac{(N-1)(N+1)}{N^2} = \frac{3}{4} \cdot \frac{N+1}{N}$$

by interior cancellation. Consequently, the indicated sequence of partial products decreases to the limit $\frac{3}{4}$ as N increases.

Next observe that

$$\prod_{n=4}^{N} \left(1 + \frac{1}{n^2}\right) = \prod_{n=4}^{N} \left(1 + \frac{1}{n^2}\right) \cdot \frac{\prod_{n=4}^{N} \left(1 - \frac{1}{n^2}\right)}{\prod_{n=4}^{N} \left(1 - \frac{1}{n^2}\right)} = \frac{\prod_{n=4}^{N} \left(1 - \frac{1}{n^4}\right)}{\prod_{n=4}^{N} \left(1 - \frac{1}{n^2}\right)} < \frac{\prod_{n=4}^{N} \left(1 - \frac{1}{n^4}\right)}{\frac{3}{4}}.$$

Each term in the product in the numerator on the right-hand side is a positive number less than 1, so the whole numerator is less than 1. Thus

$$\prod_{n=4}^{N} \left(1 + \frac{1}{n^2}\right) < \frac{4}{3},$$

so the required upper bound on the partial products has been established.