Final Examination

Part A: Sentence Completion

Your answer to each of problems 1–3 should be a complete sentence that starts as indicated.

1. The least-upper-bound property (or completeness property) of the real numbers says that if S is a non-empty set, and S is bounded above, then ....

   Solution. The least-upper-bound property (or completeness property) of the real numbers says that if S is a non-empty set, and S is bounded above, then S has a least upper bound (a supremum).

2. The mean-value theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists ....

   [Warning: Do not confuse the mean-value theorem with the intermediate-value theorem!]

   Solution. The mean-value theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in the interval (a, b) such that $f(b) - f(a) = f'(c)(b - a)$, or $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

3. A function $f : (a, b) \to \mathbb{R}$ is said to be uniformly continuous if for every positive ε ....

   Solution. A function $f : (a, b) \to \mathbb{R}$ is said to be uniformly continuous if for every positive ε, there exists a positive δ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are points in the domain of f for which $|x - y| < \delta$. (The uniformity consists in the value of δ being independent of the points x and y.)

Part B: Examples

Your task in problems 4–5 is to exhibit a concrete example satisfying the indicated property. You should provide a brief explanation of why your example works.

4. Give an example of an integrable function $f : [-1, 1] \to \mathbb{R}$ that is not differentiable at 0.

   Solution. The most popular example for $f(x)$ is $|x|$, the absolute-value function. This function is continuous on the interval $[-1, 1]$, hence integrable (by Cauchy's theorem), but the function is not differentiable when $x = 0$ (since the right-hand derivative equals 1, but the left-hand derivative equals -1.)

   Another popular example is the following function that is defined piecewise:

$$\begin{cases} 0, & \text{when } -1 \le x \le 0, \\ 1, & \text{when } 0 < x \le 1. \end{cases}$$

Since this bounded function has only one point of discontinuity, the function is integrable (by Theorem 5.2.8). Being discontinuous when $x = 0$, the function is not differentiable (invoke the contrapositive of problem 7 below).

5. Give an example of a sequence $\{x_k\}_{k=1}^{\infty}$ such that

$$\limsup_{n\to\infty} \sum_{k=1}^{n} x_k \qquad \text{and} \qquad \liminf_{n\to\infty} \sum_{k=1}^{n} x_k$$

are finite and unequal. In other words, give an example of a divergent series that has bounded partial sums.

Solution. The most popular example is to set x_k equal to $(-1)^k$. The partial sum $\sum_{k=1}^{n}(-1)^k$ is equal to -1 for odd values of n and is equal to 0 for even values of n, so $\limsup_{n\to\infty} \sum_{k=1}^{n} x_k = 0$ and $\liminf_{n\to\infty} \sum_{k=1}^{n} x_k = -1$.

Part Γ: Proof

Your proofs should be written in complete sentences, each step being justified. You may invoke theorems from the course if you indicate what the cited theorems say.

6. Suppose $x_n = \cos(e^n)$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Solution. Since the values of the cosine function lie between -1 and 1, the sequence is bounded. By the Bolzano–Weierstrass theorem, there is a convergent subsequence. [The Bolzano–Weierstrass theorem states that every bounded sequence of real numbers has a convergent subsequence.]

7. Suppose a function f is differentiable at every point of the interval $(0, 1)$. A standard proposition states that f must then be continuous at every point of the interval $(0, 1)$. Prove this proposition.

Solution. Method 1. The hypothesis implies that for an arbitrary point c in the interval,

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists and equals } f'(c).$$

Evidently $\lim_{x\to c}(x - c) = 0$. A standard theorem says that if two functions have limits, then so does the product function, and the limit of the product equals the product of the limits. Therefore

$$\lim_{x\to c} (f(x) - f(c)) = \Big(\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\Big) \cdot \lim_{x\to c}(x - c) = f'(c) \cdot 0 = 0.$$

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In other words, $\lim_{x\to c} f(x)$ exists and equals $f(c)$, which is one definition of continuity. Since the point c is arbitrary, the desired conclusion has been established.

Method 2. An alternative definition of differentiability of f at c states that there exists a function F, continuous at c, such that $f(x) = F(x)(x-c) + f(c)$. Thus $f(x)$ is the product of the two continuous functions $F(x)$ and $x - c$, plus a constant; so f is continuous at c.

Method 3. Fix a point c in the interval $(0, 1)$. Apply the definition of the derivative as a limit, with ε equal to 1, to obtain a positive number δ such that

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < 1 \qquad \text{when} \quad 0 < |x-c| < \delta \quad \text{and} \quad x \in (0,1).$$

Equivalently, $|f(x)-f(c)-f'(c)(x-c)| < |x-c|$ when $0 < |x-c| < \delta$ and $x \in (0, 1)$. The triangle inequality implies that $|f(x) - f(c)| < (|f'(c)| + 1)|x - c|$ when $0 < |x - c| < \delta$ and $x \in (0, 1)$. The squeeze theorem now implies that $\lim_{x\to c} f(x) = f(c)$, so f is continuous at an arbitrary point c.

Method 4. Argue by contradiction. Suppose there is a point c at which f is not continuous. In other words, suppose it is not the case that $\lim_{x\to c} f(x) = f(c)$. Then there exists a positive ε and a sequence $\{x_n\}_{n=1}^{\infty}$ converging to c such that $|f(x_n) - f(c)| \ge \varepsilon$. Consequently,

$$\left|\frac{f(x_n)-f(c)}{x_n-c}\right| \ge \frac{\varepsilon}{|x_n-c|}.$$

When $x_n \to c$, the fraction on the right-hand side grows without bound, so the fraction on the left-hand side cannot have a finite limit. This conclusion contradicts the hypothesis that f is differentiable at c.

Non-method. You learned a proposition stating that a function whose derivative is bounded must be a Lipschitz continuous function, hence a continuous function (and even stronger, a uniformly continuous function). This proposition is *not* applicable here, for a differentiable function need not have a *bounded* derivative. For example, if $f(x) = 2\sqrt{x}$, then f is differentiable at each point of the open interval $(0, 1)$, but the derivative equals $1/\sqrt{x}$, which is not bounded on the interval.

Part Δ: Optional Extra Credit Problem

A sequence $\{f_n\}_{n=1}^{\infty}$ of functions with common domain S is said to *converge pointwise* to a limit function f when $\lim_{n\to\infty} f_n(x)$ exists for each x in S and equals $f(x)$. Taking the domain S to be the closed interval $[0, 1]$, give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions that converges pointwise to a discontinuous limit function f.

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Solution. One example is to set $f_n(x)$ equal to x^n. The pointwise limit $f(x)$ is then equal to 0 when $0 \le x < 1$ (a standard limit contained in Proposition 2.2.11) but is equal to 1 when $x = 1$. Evidently the limit function f is discontinuous at the point 1.

Here is another example:

$$f_n(x) = \begin{cases} 1 - nx, & \text{when } 0 \le x \le 1/n, \\ 0, & \text{when } 1/n < x \le 1. \end{cases}$$

This function is continuous, for the graph consists of two line segments that join at the point where $x = 1/n$. The limit function $f(x)$ equals 0 when $0 < x \le 1$ and equals 1 when $x = 0$, hence is discontinuous at 0.