Exercise on differentiability

Recall that a function f on a domain D (a connected open set) in $\mathbb{R}^2$ is

- *continuous* (notation: $f \in C$) if for every point (a,b) in D we have $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$;
- *differentiable* if for every point (a,b) in D there exist functions φ and ψ with the property $\lim_{(x,y)\to(a,b)} \varphi(x,y) = 0 = \lim_{(x,y)\to(a,b)} \psi(x,y)$ such that we have the linear approximation formula

$$\begin{aligned} f(x,y) - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \\ + \varphi(x,y)(x-a) + \psi(x,y)(y-b); \end{aligned}$$

- *continuously differentiable* (notation: $f \in C^1$) if the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist and are continuous at all points of D.

Discuss each of the following examples (with $D = \mathbb{R}^2$) in terms of the above concepts.

A. $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

B. $g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

C. $h(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$