Modern Algebra I

This course contains many new concepts. One way to organize those concepts in your mind is to make yourself lists. Here are some suggestions about how to review for the first examination.

Definitions Make yourself a list of the main definitions. By consulting the table of contents in the textbook, you will identify the key concepts of relation, binary operation, isomorphism, group, subgroup, cyclic group, and generating set. By looking through each section, you will find various other concepts identified by the word "Definition" in boldface in the left-hand margin.

There is a list of notations in the back of the book starting on page 487. The list is arranged by first appearance of the symbol in the book. Since we have covered through page 74 of the textbook, you should recognize all the symbols on page 487 and the first half of page 488.

A definition is not much use without some examples to illustrate it. For each concept, can you give an example that satisfies the definition? Can you give an example that fails to satisfy the definition?

The next step in reviewing is to see how the definitions fit together. Typically the relations among definitions are expressed as theorems.

Theorems Make yourself a list of the main theorems. These are identified in the textbook by the word "Theorem" in boldface in the left-hand margin. You will also find theorems identified by the word "Corollary"; typically, corollaries are important special cases of more general theorems.

Theorems usually say something like, "Every X is a Y", or, "Every X has property P." Do you know examples of all the concepts mentioned in the theorem? Do you know a concrete situation to which the theorem applies? Can you paraphrase the statement of the theorem? What is the significance of the theorem?

For instance, the first theorem stated in the book is Theorem 0.22 on page 7. It says that every equivalence relation on a set induces a partition of the set, and conversely, every partition of a set induces an equivalence relation on the set. The significance of the theorem is that it says that two apparently different concepts are essentially the same. An example that you should have in mind is equivalence modulo n on the integers; the corresponding partition consists of sets of integers that leave the same remainder after division by n. The theorem implies that exercises 23–27 on page 10 of the textbook could be rephrased: instead of "find the number of different partitions of a set", the wording could be "find the number of different equivalence relations on a set"; the second wording sounds like a harder problem, but it is really the same problem in disguise.

To be sure you understand a theorem, you should also think about what the theorem does *not* say. For instance, Theorem 6.1 on page 59 says, "Every cyclic group is abelian." The theorem does not say anything about non-cyclic groups. Do there exist non-cyclic groups that are abelian? (Yes, the Klein 4-group is a non-cyclic abelian group.) In other words, the converse of the theorem is not true.

After you are sure that you understand the meaning of a theorem, and you know illustrative examples, you should ask yourself why the theorem is true. How does the proof go? Can you give a one-sentence paraphrase of the main idea in the proof? For instance, in the theorem mentioned above about equivalence relations, the idea is that an equivalence relation determines equivalence classes, and the equivalence classes form a partition of the set. In the proof of the theorem that every cyclic group is abelian, the idea is simply that different powers of the same element always commute with each other.

Examples Abstract concepts are illuminated by concrete examples. The most important notion in the first part of the course is the notion of a group. Making yourself a list of specific groups that we have encountered is a good way to review.

In particular, we know a complete list—up to isomorphism—of finite groups of small order. Groups of order 2 are all isomorphic to $\mathbb{Z}_2$. Groups of order 3 are all isomorphic to $\mathbb{Z}_3$. We know two non-isomorphic groups of order 4: namely, $\mathbb{Z}_4$ and the Klein 4-group; and every group of order 4 is isomorphic to one of these two.

[Preview of coming attractions: We will soon learn that all groups of order 5 are isomorphic to $\mathbb{Z}_5$. In particular, this means that all groups of order 5 or smaller are abelian. We will soon learn that there are two non-isomorphic groups of order 6: one is $\mathbb{Z}_6$, and the other is the group of symmetries of an equilateral triangle; the latter group is non-abelian.]

We know a complete list of finite cyclic groups: these are isomorphic to $\mathbb{Z}_n$ for some positive integer n. We also know that every infinite cyclic group is isomorphic to $\mathbb{Z}$.

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We know various examples of non-cyclic infinite groups, such as $\mathbb{Q}$ under addition, $\mathbb{Q}^+$ under multiplication, and invertible matrices under matrix multiplication. You will find other examples by looking through the book and the exercises.

Exercises You solidify your understanding of mathematics by solving problems. If there were homework exercises that you had trouble with, you should review the concepts involved. Solving additional exercises can only help improve your understanding; many of the odd-numbered exercises have answers in the back of the book, so that you can check your solutions.

About the examination As I mentioned in class, the first examination covers sections 0–7 in the textbook. There will be five true/false questions worth five points each; five fill-in-the-blanks questions worth five points each; and three essay-type questions worth fifteen points each. There will be five "style" points based on how well written your essays are; I expect that most papers will receive the style points if the solutions are clearly explained and well organized.