§1	§2	§3	style	total

Math 415 Final Examination Fall 2006

Modern Algebra I

Please print your **name**: Answer Key

1 True/false

Circle the correct answer; no explanation is required. Each problem in this section counts 5 points.

1. Every group of order 6 is cyclic. `True` `False`

   Solution. False. The symmetric group S_3 is not abelian and hence not cyclic.

2. Every element of the symmetric group S_5 can be written as a product of disjoint transpositions. `True` `False`

   Solution. False. The cycle $(1, 2, 3)$ cannot be written as a product of *disjoint* transpositions. (It can be written as the product $(1, 3)(1, 2)$, a product of non-disjoint transpositions.) What is true is that every element of the symmetric group can be written as a product of disjoint *cycles*.

3. Every subgroup of an abelian group is normal. `True` `False`

   Solution. True. A subgroup is normal if its left cosets are the same as its right cosets; if the group operation is commutative, then there is no distinction between operating on the left and operating on the right.

4. If F is a field, then the polynomial ring $F[x]$ is an integral domain. `True` `False`

   Solution. True. More generally, if R is an integral domain, then so is the polynomial ring $R[x]$.

5. The nonzero elements of a ring form a group under multiplication. `True` `False`

Solution. False. The nonzero elements of a ring need not have multiplicative inverses (example: $\mathbb{Z}$), there need not be a multiplicative identity element (example: $2\mathbb{Z}$), and the set of nonzero elements need not be closed under multiplication (example: $\mathbb{Z}_6$, where $(2)(3) = 0$).

2 Short answer

Fill in the blanks; no explanation is required. Each problem in this section counts 5 points.

6. The order of the element $(2, 3)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_8$ equals ________ .

   Solution. The order of the element 2 in $\mathbb{Z}_6$ equals 3, the order of the element 3 in $\mathbb{Z}_8$ equals 8, and the order of the element $(2, 3)$ in $\mathbb{Z}_6 \times \mathbb{Z}_8$ equals $\operatorname{lcm}(3, 8)$ or 24.

7. The permutation $(1, 2)(3, 4, 5, 6)$ generates a cyclic subgroup H of the symmetric group S_6. How many cosets does the subgroup H have?

   ____

   Solution. The given permutation generates a cyclic subgroup H of order 4. The cosets of H all have the same number of elements, and they partition the group S_6. Therefore the number of cosets is $6!/4$ or 180.

8. How many abelian groups of order 36 exist (up to isomorphism)? (Use the fundamental theorem of finite abelian groups.) __________

   Solution. Since $36 = 2^2 \times 3^2$, the possible groups (up to isomorphism) are $\mathbb{Z}_4 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus there are 4 such groups.

9. How many group homomorphisms $\phi : \mathbb{Z}_6 \to \mathbb{Z}_4$ exist? __________
   (Don't forget to count the 0 homomorphism.)

   Solution. The are 2 such homomorphisms. One way to see this is the following.

Observe that $\phi(4) = \phi(1+1+1+1) = \phi(1)+\phi(1)+\phi(1)+\phi(1) = 0$, since every element of $\mathbb{Z}_4$ added to itself 4 times produces 0 in $\mathbb{Z}_4$. Therefore the kernel of ϕ contains the element 4 and hence the subgroup of $\mathbb{Z}_6$ generated by 4: namely, $\{0, 2, 4\}$, or $\langle 2\rangle$ in different notation.

There does exist a homomorphism ϕ from $\mathbb{Z}_6$ into $\mathbb{Z}_4$ with this kernel: namely, the composition of the coset mapping $\mathbb{Z}_6 \to \mathbb{Z}_6/\langle 2\rangle \simeq \mathbb{Z}_2$ with the isomorphism from $\mathbb{Z}_2$ onto the subgroup $\{0, 2\}$ of $\mathbb{Z}_4$. This ϕ maps the elements 0, 2, and 4 of $\mathbb{Z}_6$ to the element 0 of $\mathbb{Z}_4$ and maps the elements 1, 3, and 5 of $\mathbb{Z}_6$ to the element 2 of $\mathbb{Z}_4$. By the fundamental homomorphism theorem (page 140 in the textbook), every homomorphism from $\mathbb{Z}_6$ into $\mathbb{Z}_4$ with kernel $\langle 2\rangle$ factors through $\mathbb{Z}_6/\langle 2\rangle$, and since $\mathbb{Z}_4$ has a unique subgroup of order 2, there is only one such homomorphism.

If the kernel of ϕ contains additional elements, then the kernel has to be all of $\mathbb{Z}_6$ (since the kernel is a subgroup, and by Lagrange's theorem the order of the kernel has to divide 6). In this case, ϕ is the homomorphism that sends every element to 0.

One can also solve the problem from first principles by looking at the image of the generator 1 (which uniquely determines ϕ) and showing that the homomorphism property forces $\phi(1)$ to be either 0 or 2.

10. When 3^{5000} is divided by 17 the remainder equals __________ .

Solution. Since 17 is a prime number, Fermat's theorem implies that $3^{16} \equiv 1 \mod 17$. Now $5000 = (312 \times 16) + 8$, so $3^{5000} \equiv 3^8 \mod 17$. Finally observe that $3^8 = 81^2 \equiv (-4)^2 \equiv 16 \mod 17$. Thus the remainder equals 16.

3 Essay questions

In the following problems, you must give an explanation. (Continue on the back if you need more space.) Each problem counts 15 points. In addition, this section as a whole carries 5 style points based on how well your solutions are written.

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11. Give an example of a group G that has a pair of subgroups H and K with the property that the union $H \cup K$ is not a subgroup. Explain why your example works.

    Solution. What can go wrong is that the union $H \cup K$ may not be closed under the group operation.

    There are many examples. Here is a popular one: $G = \mathbb{Z}_6$, $H = \{0, 2, 4\}$, $K = \{0, 3\}$. Then H and K are subgroups of G, but the union $\{0, 2, 3, 4\}$ is not a subgroup, because it contains the elements 2 and 3 but not the sum $2 + 3$.

12. Suppose G is a simple group (recall this means that G has no proper, nontrivial, normal subgroup), and $\phi : G \to G'$ is a homomorphism onto a nontrivial group G'. Show that ϕ is an isomorphism.

    Solution. An isomorphism is a homomorphism that is both one-to-one and onto. It is given that the homomorphism ϕ is onto, so what needs to be shown is that ϕ is one-to-one. In other words, we need to show that the kernel of ϕ is trivial.

    Now the kernel of ϕ is a normal subgroup of G; since G is simple, $\mathrm{Ker}(\phi)$ must be either trivial or all of G. In the second case, ϕ would send every element of G to the identity element of G', so ϕ would not map onto the nontrivial group G', contrary to hypothesis. Thus the only possibility is that the kernel of ϕ is the trivial subgroup of G, and so ϕ is one-to-one.

13. Show that the polynomial $x^3 + 4x^2 + 4x + 2$ is irreducible over $\mathbb{Q}$ but factors into linear factors over the field $\mathbb{Z}_{11}$.

    Solution. The prime number 2 divides each of the coefficients except the leading coefficient, and 2^2 does not divide the constant coefficient. Therefore Eisenstein's criterion applies and shows that the polynomial is irreducible over $\mathbb{Q}$.

    Alternatively, observe that since the polynomial has degree 3, it factors only if it has at least one factor of degree 1, or equivalently if it has at least one zero. The only candidates for zeroes in $\mathbb{Q}$ are ± 1 and ± 2.

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One easily checks that none of the candidates actually is a zero, so the polynomial is irreducible over $\mathbb{Q}$.

By inspection, 1 is a zero of the polynomial in the field $\mathbb{Z}_{11}$, since $1+4+4+2=11$. Therefore $(x-1)$, or equivalently $(x+10)$, is a factor of the polynomial over this field. Long division in $\mathbb{Z}_{11}[x]$ shows that $x^3+4x^2+4x+2=(x+10)(x^2+5x+9)$ in $\mathbb{Z}_{11}[x]$. Now 5 is the same as $-6 \pmod{11}$, and $x^2-6x+9=(x-3)^2$, so x^3+4x^2+4x+2 factors over $\mathbb{Z}_{11}$ as $(x+10)(x+8)^2$.