

Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

- You know that \mathbb{Z} (the set of integers) is an abelian group under addition. You know too that \mathbb{Z}_2 (the integers modulo 2) is a field (since the number 2 is prime). Consider the “scalar multiplication” operation of \mathbb{Z}_2 on \mathbb{Z} defined as follows: $[0]_2 \cdot n = 0$ and $[1]_2 \cdot n = n$ for every integer n . Does this operation give \mathbb{Z} the structure of a vector space over the field \mathbb{Z}_2 ? Explain why or why not.

Solution. Here are the four properties that need to be checked (see page 191 in the textbook):

- $(\lambda\mu) \cdot n = \lambda \cdot (\mu \cdot n)$ for every integer n and for all λ and μ in \mathbb{Z}_2 ;
- $[1]_2 \cdot n = n$ for every integer n [this property is given];
- $(\lambda + \mu) \cdot n = \lambda \cdot n + \mu \cdot n$ for every integer n and for all λ and μ in \mathbb{Z}_2 ;
- $\lambda \cdot (n + k) = \lambda \cdot n + \lambda \cdot k$ for every λ in \mathbb{Z}_2 and for all integers n and k .

The first, second, and fourth properties do hold, but the third property fails. Indeed, $([1]_2 + [1]_2) \cdot n \neq [1]_2 \cdot n + [1]_2 \cdot n$, for the left-hand side simplifies to $[0]_2 \cdot n$ or 0, while the right-hand side simplifies to $n + n$ or $2n$; when $n \neq 0$, the two sides fail to match. Therefore the indicated structure is not a vector space over the field \mathbb{Z}_2 .

- The two partially completed tables shown are the addition table and the multiplication table for a certain field. The labels 0 and 1 represent the additive identity element and the multiplicative identity element. Fill in the missing entries.

+	0	1	a	b
0	0	1	a	b
1	1	0		
a	a			
b	b			

\cdot	0	1	a	b
0		0		
1	0	1	a	b
a		a		
b		b		

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Solution. Here are the completed tables.

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

\cdot	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Here is some explanation about to get this answer.

First look at the addition table. Each element has to appear exactly once in each row and in each column of this group table. Hence the 1-row has to be completed with b and a in that order (otherwise there would be two copies of b in the last column). Now you can fill in the 1-column by symmetry (since addition in a field is commutative). The table shows that $1 + 1 = 0$, so multiplying by a reveals that $a + a = 0$; similarly, $b + b = 0$. Thus both missing entries on the main diagonal are 0. Both remaining blanks must be filled with the entry 1 to have a copy of 1 in each row.

In the multiplication table, the 0-row and the 0-column must have all entries equal to 0. (We proved a proposition stating that $0 \cdot x = 0$ in every ring.) The nonzero elements of a field form a group under multiplication, so the 3×3 block in the lower right-hand corner of the table must have exactly one instance of each of 1, a , and b in each row and in each column. Therefore the values of the remaining four entries are forced.

Remark This field structure on a set of four elements is unique (up to isomorphism).