Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Consider the set of integers that are divisible either by 2 or by 3 or by both 2 and 3. Is this sub*set* of the integers a sub*group* of the integers (under addition)? Explain why or why not.

   Solution. The set is not a subgroup, for the set is not closed under addition: the numbers 2 and 3 are in the set, but their sum 5 is not in the set.

2. List all the cyclic subgroups of the group $(\mathbb{Z}_6, +)$ [the integers mod 6 under addition].

   Solution. Remark on notation: The elements of $\mathbb{Z}_6$ are the equivalence classes $[0]_6$, $[1]_6$, $[2]_6$, $[3]_6$, $[4]_6$, and $[5]_6$, but it is reasonable in the context of this problem to use the representatives 0, 1, 2, 3, 4, and 5 as abbreviations for the equivalence classes.

   The element 0 generates the cyclic subgroup consisting of the singleton $\{0\}$.

   The element 1 generates the whole cyclic group $\mathbb{Z}_6$. The element 5 is an alternative generator of $\mathbb{Z}_6$.

   The element 2 generates the cyclic subgroup $\{0, 2, 4\}$. The element 4 is an alternative generator of this subgroup.

   The element 3 generates the cyclic subgroup $\{0, 3\}$.

   Remark It is a theorem that every subgroup of a cyclic group is necessarily cyclic. See Exercise 7 on page 212 in section 5.1.