Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. List the elements of the alternating group $A(3)$ and state the order of each element.

   Solution. The symmetric group $S(3)$ has six (since $3! = 6$) elements. Three of these are odd permutations (namely, the transpositions $(1\ 2)$, $(2\ 3)$, and $(1\ 3)$).

   The alternating group $A(3)$ consists of the even permutations in $S(3)$: namely, the remaining three. These are the identity permutation, which has order 1; the cycle $(1\ 2\ 3)$, which has order 3, and the cycle $(1\ 3\ 2)$, which has order 3.

2. Explain why every odd permutation in the symmetric group $S(n)$ must have even order.

   Solution. *First method.* If π is an odd permutation, then π can be written as the product of an odd number of transpositions, say $2j + 1$ transpositions. Then a power π^k can be written as a product of $(2j + 1)k$ transpositions (simply by writing down the product of $2j + 1$ transpositions k times). Therefore π^k is an odd permutation if k is odd and an even permutation if k is even (since the product $(2j+1)k$ is odd if k is odd and even if k is even). Since the identity permutation is even, the power π^k can never equal the identity when k is odd. Therefore the order of π (the smallest k such that π^k equals the identity) must be even.

   Second method. You know that a cycle of length k is an odd permutation if k is even and an even permutation if k is odd. You also know that the order of a product of disjoint cycles is the least common multiple of their lengths. Suppose an odd permutation π is written as a product of disjoint cycles. The lengths of these cycles cannot all be odd, for then π would be a product of even permutations and hence would be even (which it is not). Since at least one of the cycles has even length, the least common multiple of their lengths must be even. Thus the order of π is even.