Exercise on the unit circle

Exercise 3(a) on page 239 asks for an intuitive argument showing that the identity function on the unit circle is not homotopic to a constant function. The goal today is to find a rigorous justification, and also to generalize.

Preliminaries Suppose that $\mathbb{R}$ and $\mathbb{R}^2$ carry the standard topology. Equip the unit circle S, the set $\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$, with the subspace topology inherited from $\mathbb{R}^2$. Let I denote the unit interval, $\{ x \in \mathbb{R} : 0 \le x \le 1 \}$, equipped with the standard topology.

Your task is to justify the following statements.

Covering map There is a function $p\colon \mathbb{R} \to S$ that is continuous, surjective, and *locally* injective. Specifically, if $p(t) = (\cos(2\pi t), \sin(2\pi t))$, then the function p has the indicated properties.

The reason for the factor of 2π is that this normalization makes the integers map to the point $(1, 0)$ of the circle. Moreover, the function p is periodic with period 1. If you are familiar with $\mathbb{C}$, the complex numbers, then you can think of $p(t)$ as being $\exp(2\pi i t)$ under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$.

Path lifting If $f : I \to S$ is a path (a continuous function) with initial point $f(0)$ equal to $(1, 0)$, then there exists a unique path $\widehat{f}\colon I \to \mathbb{R}$ with initial point $\widehat{f}(0)$ equal to 0 such that $p \circ \widehat{f} = f$.

Degree of a loop A path $f : I \to S$ is a *loop* with base point $(1, 0)$ if $f(0)$ and $f(1)$ both equal the point $(1, 0)$. By the preceding paragraph, such a loop has a unique lifting to a path $\widehat{f}\colon I \to \mathbb{R}$ for which $\widehat{f}(0) = 0$. Then $\widehat{f}(1)$ is equal to some integer, which is called the *degree* of the loop f.

In complex analysis, this concept is called the *winding number*. The degree measures the number of times that the loop winds around the unit circle.

Homotopy classes of loops By definition, two loops f and g with the same base point $(1, 0)$ are homotopic *relative to* $(1, 0)$ if there exists a continuous function $H : I \times I \to S$ such that $H(t, 0) = f(t)$ for every t and $H(t, 1) = g(t)$ for every t and $H(0, s) = H(1, s) = (1, 0)$ for every s. (The latter condition means that all of the intermediate loops in the homotopy have the same base point.)

Prove that two loops are homotopic relative to the base point $(1, 0)$ if and only if these loops have equal degrees.