Examination 1

1. Suppose $X = \{a, b\}$. List all possible topologies on X.

2. Define the following concepts:
   (a) interior of a set, and
   (b) basis for a topology.

3. Explain why $\{ (x, y) \in \mathbb{R}^2 : xy = 0 \}$ is a closed subset of the metric space $\mathbb{R}^2$ with the usual Pythagorean metric.

4. Consider $\mathbb{Q}$ (the set of rational numbers) as a subset of $\mathbb{R}$ (the real numbers). Determine the frontier (the boundary) of $\mathbb{Q}$
   (a) when the metric on $\mathbb{R}$ is the usual absolute-value metric, and
   (b) when the metric on $\mathbb{R}$ is the discrete metric.
   [Recall that the discrete metric is the metric for which the distance between every two different points is equal to 1.]

5. Let $d(x, y)$ denote $\log(1 + |x - y|)$ for real numbers x and y. Show that d is a metric on $\mathbb{R}$. [Reminder: the characteristic property of logarithms says that $\log(u) + \log(v) = \log(uv)$.]

6. True or false: If $\mathbb{R}$ is equipped with the discrete metric, then every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous. Explain your answer.

7. Suppose τ is a topology on $\mathbb{R}^2$ with the property that every line is a τ-open set. Prove that τ must be the discrete topology (the topology in which every subset of $\mathbb{R}^2$ is open).

8. Suppose τ is a topology on a set X. Let σ be the collection of all τ-closed subsets of X. Is this collection σ a topology on X? Explain why or why not.

Extra Credit. Valentine's Day Bonus Problem:

Suppose X is a topological space. Define $\heartsuit A$ to be $(A')^\circ$ (that is, the interior of the derived set of A) when $A \subset X$. Prove that $\heartsuit(\heartsuit A) = \heartsuit A$.