Examination 1

1. Suppose $X = \{a, b\}$. List all possible topologies on X.

   Solution. Every topology must contain at least the empty set and the whole set. The only other subsets of X are the singletons $\{a\}$ and $\{b\}$. A topology can contain neither of these singletons, either one of them, or both of them. Therefore the possible topologies on X are the following:

$$\{\varnothing, \{a, b\}\} \tag{1}$$
$$\{\varnothing, \{a\}, \{a, b\}\} \tag{2}$$
$$\{\varnothing, \{b\}, \{a, b\}\} \tag{3}$$
$$\{\varnothing, \{a\}, \{b\}, \{a, b\}\}. \tag{4}$$

   In principle, you should check that all four cases truly are topologies: namely, closed under unions and under finite intersections. But the sets are so simple that this conclusion is evident by inspection.

   If you really want a formal proof that each collection is closed under unions and under finite intersections, then you could argue as follows. Case (4) is the discrete topology, consisting of all subsets of X, so there is nothing to check. The other three cases all are collections of sets that can be linearly ordered by inclusion. For any finite chain of sets ordered by inclusion, the union is the largest of the sets, and the intersection is the smallest of the sets. Therefore no new sets can arise from forming unions or intersections.

2. Define the following concepts:

   (a) interior of a set, and

   (b) basis for a topology.

   Solution. The interior of a set is the union of all open sets contained in it (Definition 7 on page 55 in Section 3.5). In other words, the interior of A is the largest open subset of A.

   A basis for a topology τ is a collection of τ-open sets such that every τ-open set can be obtained by taking the union of some of the sets in the collection (Definition 3 on page 44 in Section 3.2).

3. Explain why $\{\, (x, y) \in \mathbb{R}^2 : xy = 0 \,\}$ is a closed subset of the metric space $\mathbb{R}^2$ with the usual Pythagorean metric.

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Solution. The indicated set is the union of the two coordinate axes.

One way to see that a set is closed is to show that the complement is open. If (a, b) is a point in the complement, then neither a nor b is equal to zero, and the neighborhood of the point (a, b) of radius equal to $\min(|a|, |b|)$ does not intersect the coordinate axes. Accordingly, the complement of the coordinate axes contains a neighborhood of each of its points, hence is open.

Another way to see that the given set is closed is to recall from calculus that the function xy is continuous with respect to the standard metric. Singleton sets are closed in metric spaces, so the given set is closed because it is the inverse image of the closed set $\{0\}$ under a continuous function.

4. Consider $\mathbb{Q}$ (the set of rational numbers) as a subset of $\mathbb{R}$ (the real numbers). Determine the frontier (the boundary) of $\mathbb{Q}$

   (a) when the metric on $\mathbb{R}$ is the usual absolute-value metric, and

   (b) when the metric on $\mathbb{R}$ is the discrete metric.
   [Recall that the discrete metric is the metric for which the distance between every two different points is equal to 1.]

Solution.

(a) With respect to the standard metric, $\operatorname{Fr}\mathbb{Q} = \mathbb{R}$. Indeed, every interval in $\mathbb{R}$ contains both rational numbers and irrational numbers. In other words, every neighborhood of an arbitrary point intersects both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. Therefore every point is in the frontier of $\mathbb{Q}$.

An alternative argument is that the interior of $\mathbb{Q}$ is empty (since $\mathbb{Q}$ contains no interval), and the closure of $\mathbb{Q}$ is $\mathbb{R}$ (since $\mathbb{R} \setminus \mathbb{Q}$ contains no interval). Then $\operatorname{Fr}\mathbb{Q} = (\operatorname{Cl}\mathbb{Q}) \setminus \mathbb{Q}^\circ = \mathbb{R} \setminus \varnothing = \mathbb{R}$.

A third solution is to observe that the rational numbers are dense in the real numbers, and the irrational numbers are dense too, so $\operatorname{Fr}\mathbb{Q} = (\operatorname{Cl}\mathbb{Q}) \cap \operatorname{Cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

(b) With respect to the discrete metric, $\operatorname{Fr}\mathbb{Q} = \varnothing$. Indeed, singleton sets are open in the discrete metric, so every point x has a neighborhood, namely the singleton $\{x\}$, that intersects either $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ but not both. Hence no point x satisfies the condition to be in the frontier of $\mathbb{Q}$.

An alternative argument is that the interior of $\mathbb{Q}$ is equal to $\mathbb{Q}$ (since every set is open with respect to the discrete metric), and the closure of $\mathbb{Q}$ is equal to $\mathbb{Q}$ (since every set is closed with respect to the discrete metric). Then $\operatorname{Fr}\mathbb{Q} = (\operatorname{Cl}\mathbb{Q}) \setminus \mathbb{Q}^\circ = \mathbb{Q} \setminus \mathbb{Q} = \varnothing$.

Actually, a similar argument shows that *every* set has empty frontier with respect to the discrete metric.

5. Let $d(x, y)$ denote $\log(1 + |x - y|)$ for real numbers x and y. Show that d is a metric on $\mathbb{R}$. [Reminder: the characteristic property of logarithms says that $\log(u) + \log(v) = \log(uv)$.]

Solution. Some of the properties of a metric are easy to check. Symmetry holds because $|x - y| = |y - x|$. The property that $d(x, x) = 0$ for every x is valid because $\log(1) = 0$. The property that $d(x, y) > 0$ when $x \neq y$ follows because the logarithm function is increasing, so $\log(1+|x-y|) > \log(1) = 0$.

What remains to verify is the triangle inequality for d: namely,

$$\log(1 + |x - y|) \overset{?}{\leq} \log(1 + |x - z|) + \log(1 + |z - y|). \tag{5}$$

A popular way to check this inequality is to manipulate it into some other inequality that is obviously true. That method is acceptable but prone to error (because making a step that is not reversible will wreck the proof).

Perhaps safer is to argue by contradiction. Suppose, if possible, that there are values of x, y, and z for which inequality (5) fails: namely,

$$\log(1 + |x - y|) > \log(1 + |x - z|) + \log(1 + |z - y|). \tag{6}$$

The exponential function is strictly increasing, so exponentiating preserves the inequality:

$$1 + |x - y| > (1 + |x - z|)(1 + |z - y|) = 1 + |x - z| + |z - y| + |x - z|\,|z - y|.$$

Subtract 1 from both sides to see that

$$|x - y| > |x - z| + |z - y| + |x - z|\,|z - y|.$$

Since $|x-z|\,|z-y| \geq 0$, the points x, y, and z have the property that $|x-y| > |x - z| + |z - y|$, which violates the triangle inequality for the absolute-value metric. This contradiction shows that inequality (6) is untenable, so the required inequality (5) holds after all.

Remark. The same argument shows that if D is an arbitrary metric, then $\log(1 + D)$ is a metric too.

6. True or false: If $\mathbb{R}$ is equipped with the discrete metric, then every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous. Explain your answer.

   Solution. True. A function is continuous if and only if the inverse image of every open set is open. Every set is open in the discrete metric, so, in particular, every inverse-image set is open.

   An alternative explanation is that continuous functions are the functions that map convergent sequences to convergent sequences. A sequence is convergent with respect to the discrete metric precisely when the sequence is eventually constant. An arbitrary function evidently maps every eventually constant sequence to an eventually constant sequence.

   Remark. Example 7 on page 70 in Section 4.3 is a generalization of this problem to the setting of topological spaces.

7. Suppose τ is a topology on $\mathbb{R}^2$ with the property that every line is a τ-open set. Prove that τ must be the discrete topology (the topology in which every subset of $\mathbb{R}^2$ is open).

   Solution. The intersection of two open sets is open, so the intersection of every pair of lines is τ-open. Every point can be obtained as the intersection of two lines, so every point (that is, every singleton set) is τ-open. Every set is a union of points, so every set is a union of τ-open sets. Hence every set is τ-open. Thus τ is the discrete topology.

8. Suppose τ is a topology on a set X. Let σ be the collection of all τ-closed subsets of X. Is this collection σ a topology on X? Explain why or why not.

   Solution. Sometimes σ is a topology, but not always. For instance, if τ is the discrete topology consisting of all subsets of X, then σ is equal to τ, so σ is a topology. On the other hand, if X is a metric space whose topology τ is not the discrete topology (for example, the set of real numbers with the standard absolute-value metric), then σ is not a topology. Indeed, singleton sets are closed sets in metric spaces, so σ contains all singletons. If σ were a topology, then the union of sets in σ would be in σ. But every set is a union of singleton sets, so every set would be in σ. Thus σ would be the discrete topology, so τ too would be the discrete topology, contrary to hypothesis.

The underlying subtlety here is that *arbitrary* unions of open sets are open and *finite* intersections of open sets are open. (Recall Exercise 1 on page 43 in Section 3.1.) Closed sets are complements of open sets, so DeMorgan's laws for set complements imply that arbitrary intersections of sets in σ still belong to σ and finite unions of sets in σ still belong to σ. Since arbitrary unions of sets in σ do not necessarily belong to σ, the collection σ need not be a topology. But if there are only finitely many τ-open sets (which is certainly the case if X is a finite set), then σ will be a topology.

Extra Credit. Valentine's Day Bonus Problem:

Suppose X is a topological space. Define $\heartsuit A$ to be $(A')^\circ$ (that is, the interior of the derived set of A) when $A \subset X$. Prove that $\heartsuit(\heartsuit A) = \heartsuit A$.

Solution. The strategy is to show both that $\heartsuit A$ is a subset of $\heartsuit(\heartsuit A)$ and that $\heartsuit(\heartsuit A)$ is a subset of $\heartsuit A$. Notice that if a set B is open, then B is a subset of C if and only if B is a subset of the interior of C. And the sets $\heartsuit A$ and $\heartsuit(\heartsuit A)$ are open, since they are interiors of other sets. Therefore all that needs to be shown is that $\heartsuit A \subset (\heartsuit A)'$ and $\heartsuit(\heartsuit A) \subset A'$.

For the first step, let x be an arbitrary point of $\heartsuit A$. The goal is to show that for every neighborhood U of x, the intersection $U \cap \heartsuit A$ contains some point different from x. Seeking a contradiction, suppose that $U \cap \heartsuit A = \{x\}$. Then the singleton set $\{x\}$ is open, being the intersection of the open sets U and $\heartsuit A$. But now x cannot belong to A', since $\{x\}$ is a neighborhood of x that does not intersect $A \setminus \{x\}$. This conclusion contradicts that x lies in $\heartsuit A$, which is a subset of A'. The contradiction means that every neighborhood U of x does intersect $(\heartsuit A) \setminus \{x\}$, so $x \in (\heartsuit A)'$. Therefore $\heartsuit A \subset (\heartsuit A)'$, since x is an arbitrary point of $\heartsuit A$.

For the second step, let x be an arbitrary point of $\heartsuit(\heartsuit A)$. The goal is to show that for every neighborhood U of x, the intersection $U \cap A$ contains some point different from x. This conclusion is automatic if $x \in \heartsuit A$, since $\heartsuit A$ is a subset of A', so suppose that $x \notin \heartsuit A$. By hypothesis, x lies in the derived set of $\heartsuit A$, so the neighborhood U intersects $\heartsuit A$. Let y be a point in $U \cap \heartsuit A$. The set $U \cap \heartsuit A$ is open, hence is a neighborhood of y, and $y \in A'$. Therefore the neighborhood $U \cap \heartsuit A$ intersects A, and necessarily at some point different from x, since $x \notin \heartsuit A$.

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Accordingly, every neighborhood U of x intersects $A \setminus \{x\}$, so $x \in A'$. Therefore $\heartsuit(\heartsuit A) \subset A'$, since x is an arbitrary point of $\heartsuit(\heartsuit A)$.

In conclusion, the preceding two steps show that $\heartsuit A \subset \heartsuit(\heartsuit A) \subset \heartsuit A$. Therefore $\heartsuit A = \heartsuit(\heartsuit A)$.

The heart has its reasons that Reason knows not.[1]
— Blaise Pascal (1623–1662)

[1] Le cœur a ses raisons, que la raison ne connaît point.