Solution to Exercise 1 in Section 3.1

The exercise asks for a proof that (a) the intersection of finitely many open sets is open, and (b) the union of finitely many closed sets is closed. The natural method of proof for this problem is the method of mathematical induction from Math 220.

To prove part (a), let P_n be the statement, "If each of the sets $U_1, \ldots, U_n$ is open, then the intersection $U_1 \cap \cdots \cap U_n$ is open." The goal is to prove that statement P_n is valid for every natural number n.

For the basis step ($n = 1$), observe that the intersection of a single open set is simply the set itself, hence is open. Thus statement P_1 holds trivially.

The induction step requires showing for an arbitrary natural number n that if statement P_n holds, then statement P_{n+1} holds. Suppose, then, that n is a natural number for which statement P_n holds. To prove that statement P_{n+1} follows as a consequence, consider an arbitrary collection of open sets $U_1, \ldots, U_{n+1}$, and observe that the intersection of these sets can be expressed as

$$\bigl(U_1 \cap \cdots \cap U_n\bigr) \cap U_{n+1} \tag{1}$$

(by the definition of intersection). By the induction hypothesis P_n, the set $U_1 \cap \cdots \cap U_n$ is open. Thus expression (1) is the intersection of the two open sets $U_1 \cap \cdots \cap U_n$ and U_{n+1}. According to Definition 1(ii) on page 40 (the definition of a topology), the intersection of two open sets is open, so (1) is open. Thus the intersection of the open sets $U_1, \ldots, U_{n+1}$ is open, and this conclusion demonstrates the validity of the implication that $P_n \implies P_{n+1}$.

Since both the basis step and the induction step hold, the method of mathematical induction implies that statement P_n holds for every natural number n.

Part (b) can similarly be proved by the method of mathematical induction, but a simpler method is to deduce part (b) as a corollary of part (a) by applying DeMorgan's laws (which are reviewed in Proposition 1 on page 2). Namely, if the sets $F_1, \ldots, F_n$ are closed subsets of a topological space X, then the complementary sets $X \setminus F_1, \ldots, X \setminus F_n$ are open sets. By part (a), the intersection

$$(X \setminus F_1) \cap \cdots \cap (X \setminus F_n) \tag{2}$$

is an open set. Therefore the complement of (2) is a closed set. But the complement of an intersection is the union of the complements, so the set $F_1 \cup \cdots \cup F_n$ is closed. This conclusion establishes part (b).