Examination 1

1. Define
   (a) the concept of the closure of a set, and
   (b) the concept of a basis for a given topology.

2. Give an example of a separable Hausdorff topological space (X, τ) with the property that X has no limit points.

3. Suppose (X, τ) is a topological space, and Y is a dense open subset equipped with the subspace topology. If the space X is connected, must Y be connected too? Provide a proof or a counterexample, whichever is appropriate.

4. The textbook defines a topological space to be a T_2 space if every two distinct points admit disjoint open neighborhoods. Prove that this property is equivalent to the following: for every two distinct points x and y, there exists an open set V such that $x \in V$ and $y \notin \overline{V}$ (where $\overline{V}$ denotes the closure of V).

5. Suppose (X_1, τ_1) is the set of rational numbers equipped with the usual topology induced by the Euclidean topology on $\mathbb{R}$, and (X_2, τ_2) is the set of natural numbers equipped with the usual topology induced by the Euclidean topology on $\mathbb{R}$. Are the topological spaces (X_1, τ_1) and (X_2, τ_2) homeomorphic to each other? Explain why or why not.

Optional Extra Credit Problem

Notation. Let (X, τ) be a topological space. When A is a subset of X, let A^- denote the closure of A, and let A^c denote the complement of A (that is, $A^c = X \setminus A$).

Problem. Let A be a subset of X. Prove that A is equal to the closure of some open set if and only if $A = A^{c-c-}$ (the closure of the complement of the closure of the complement).

Historical note. The source for this statement is the doctoral thesis of the renowned Polish mathematician Kazimierz (Casimir) Kuratowski (1896–1980). Published as "Sur l'opération $\overline{A}$ de l'Analysis Situs" [*Fundamenta Mathematicae* **3** (1922) 182–199], the first part of Kuratowski's dissertation develops topology axiomatically based on the closure operation. Kuratowski deduced from the indicated statement that by repeatedly applying the operations of forming complements and forming closures, it is possible to create from a given set A at most 14 distinct sets (including the original set A).