Examination 2

1. Equip $\mathbb{R}$, the set of real numbers, with the standard topology (corresponding to the absolute-value metric). Let the subspace Y be the half-open interval $[0, 2)$, that is, $\{ x \in \mathbb{R} : 0 \leq x < 2 \}$. Let A be the interval $[0, 1)$. With respect to the subspace topology on Y, is the set A open, closed, both, or neither? Explain.

2. Equip $\mathbb{N}$, the set of natural numbers, with the cofinite topology (that is, the proper closed sets are the finite sets). With respect to the corresponding product topology on the product space $\mathbb{N} \times \mathbb{N}$, is the "diagonal" subset

$$\{ (n, n) \in \mathbb{N} \times \mathbb{N} : n \in \mathbb{N} \}$$

open, closed, both, or neither? Explain.

3. Let X be $\{ x \in \mathbb{R} : 0 < x \}$ (the set of positive real numbers) equipped with the discrete topology, and let $f : X \to X$ be defined by setting $f(x)$ equal to x^2 for each value of x. Is this function f a homeomorphism? Explain why or why not.

4. Prove that if a topological space satisfies the separation property T_4 and also satisfies the separation property T_1, then the space necessarily satisfies the separation property T_2.

5. State either Urysohn's Lemma or Tietze's Extension Theorem.

6. Does there exist a topology τ on the set $\mathbb{R}$ of real numbers that makes $(\mathbb{R}, \tau)$ into a *compact* topological space? Explain why or why not.

7. With respect to the standard topology on the real numbers, there does *not* exist a function $f : [0, 1] \to (0, 1)$ that is simultaneously continuous and surjective (onto). Why not?

8. Give an example of a topological space that is first countable but not second countable. Explain why your example works.

Bonus Problem for Extra Credit:

Prove the following version of Cantor's nested-set theorem: If X is a Hausdorff topological space, and $\{K_j\}_{j=1}^{\infty}$ is a decreasing sequence of nonempty compact subsets of X (that is, $K_1 \supset K_2 \supset \cdots$), then the intersection $\bigcap_{j=1}^{\infty} K_j$ is not empty.