Final Examination

1. Suppose $X = \{1, 2, 3\}$, and $\mathcal{B} = \{\{1, 2\}, \{2, 3\}\}$. Is the set $\mathcal{B}$
   (a) a topology on X?
   (b) a basis for a topology on X?
   (c) a subbasis for a topology on X?

   Explain why or why not.

2. Suppose X is the set $\{x, y, z\}$ equipped with the topology $\{\varnothing, \{x\}, \{x, y\}, X\}$. Determine
   (a) the interior of the singleton subset $\{y\}$, and
   (b) the closure of the singleton subset $\{y\}$.

3. Suppose $d(x, y) = \left|\log\left(\frac{x}{y}\right)\right|$. Is this function d a metric on the set of positive rational numbers? Explain why or why not. [Recall that $\log(x/y) = \log(x) - \log(y)$.]

4. The topological space in this problem is $\mathbb{R}$, the set of real numbers, equipped with the standard Euclidean topology.
   (a) Give an example of a subset of $\mathbb{R}$ that is compact but not connected.
   (b) Give an example of a subset of $\mathbb{R}$ that is connected but not compact.

5. Suppose $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is the function defined as follows: $f(x) = \begin{cases} 1/x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$
   (a) Give an example of a topology τ with respect to which the function f is continuous.
   (b) Give an example of a topology τ with respect to which f is not continuous.

6. True/false: For each of the following statements, say whether the sentence is true or false (exclusive "or"). If the statement is false, give a counterexample; if the statement is true, give a brief explanation why.
   (a) Every path-connected topological space is connected.
   (b) Every metric space is a Hausdorff space (that is, T_2 space) with respect to the topology induced by the metric.
   (c) A subset E of a topological space X is dense in X if and only if the following property holds: $E \cap U \neq \varnothing$ for every nonempty open set U.

Optional Extra Credit Problem

Determine the homeomorphism classes of intervals in $\mathbb{R}$ with respect to the Sorgenfrey topology. In other words, if intervals are equipped with the subspace topology induced by the Sorgenfrey topology on $\mathbb{R}$, then which intervals are homeomorphic to each other?