MATH 436 **Final Exam** **14 December 1993**

1. (This problem has 9 parts worth 4 points each.) The interval $[0, 1]$ is a subset of the real numbers $\mathbb{R}$, and so $[0, 1]$ may be viewed as a topological space with the relative topology it inherits from $\mathbb{R}$. When the topology on $\mathbb{R}$ is (a) the usual $\mathcal{U}$ topology, (b) the open half-line $\mathcal{C}$ topology, (c) the half-open interval $\mathcal{H}$ topology, state whether or not $[0, 1]$ is (i) a compact topological space, (ii) a connected topological space, (iii) a Hausdorff topological space. In other words, fill in the nine entries in the table with "Yes" or "No" as appropriate:

$[0, 1]$	compact	connected	Hausdorff
$\mathcal{U}$			
$\mathcal{C}$			
$\mathcal{H}$			

2. (This problem has 3 parts worth 7 points each.) Consider the real numbers $\mathbb{R}$ with the usual $\mathcal{U}$ topology. Give an example of
   (a) a closed, infinite subset of $\mathbb{R}$ with empty interior;
   (b) a compact, disconnected subset of $\mathbb{R}$;
   (c) a dense subset of $\mathbb{R}$ whose complement is also a dense subset of $\mathbb{R}$.

3. (7 points) Give a precise statement of any *one* of the following four theorems:
   (i) DeMorgan's Laws
   (ii) Heine-Borel Theorem
   (iii) Urysohn's Lemma
   (iv) Tychonoff's Theorem

The remaining 6 problems count 6 points each. For each question, give a brief explanation if the answer is "Yes", and supply a counterexample if the answer is "No".

4. If X is a Hausdorff topological space, Y is a metric space, and $f : X \to Y$ is a continuous function, must $f(U)$ be an open subset of Y whenever U is an open subset of X?

5. Is $\operatorname{Int}(\operatorname{Cl}(A)) = A$ for every subset A of a topological space?

6. If X is a compact topological space and $f : X \to \mathbb{R}$ is a continuous function (where $\mathbb{R}$ has the usual $\mathcal{U}$ topology), must the image $f(X)$ be a bounded interval?

7. Let X and Y be topological spaces, and suppose the Cartesian product $X \times Y$ has the standard product topology. If A is a subset of X and B is a subset of Y, must it be the case that $\operatorname{Bd}(A \times B) = \operatorname{Bd}(A) \times \operatorname{Bd}(B)$?

8. Suppose X and Y are topological spaces, and $f : X \to Y$ is a continuous function. If B is a connected subset of Y, must the inverse image $f^{-1}(B)$ be a connected subset of X?

9. If $(X, \mathcal{T})$ is a compact, connected topological space, and $\mathcal{S}$ is a coarser topology than $\mathcal{T}$ (that is, $\mathcal{S} \subseteq \mathcal{T}$), must the topological space $(X, \mathcal{S})$ also be compact and connected?

Extra credit (6 points)

10. Prove that a compact Hausdorff space is necessarily a T_4 space.