Continuity is a one-way street

Continuous function: the *inverse* image of every open set is open.

But the *direct* image of an open set is not necessarily open.

Example

Consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. The open interval $(-1, 1)$ in the domain has image $[0, 1)$, which is not an open subset of $\mathbb{R}$.

Definition

A function is called *open* if the image of every open set is open.

Why did the chicken cross the road?

Theorem (Intermediate-value theorem from calculus)

If I is an interval, and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.

[Euclidean topology is assumed.]

Theorem (Generalization to topological spaces)

If $f : X_1 \to X_2$ is continuous [where (X_1, τ_1) and (X_2, τ_2) are two topological spaces], and A is a connected subspace of X_1, then $f(A)$ is a connected subspace of X_2.

Path-connected spaces

A *path* joining a to b in a space X means a continuous function $f\colon [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$.

A space is *path-connected* when each two points in the space can be joined by a path.

Example

$\mathbb{R}^2$ is path-connected. Two arbitrary points (x_1, y_1) and (x_2, y_2) can be joined by a straight-line path: namely,

$$f(t) = ((1-t)x_1 + tx_2, (1-t)y_1 + ty_2).$$

Path-connected $\implies$ connected

. . . to be continued

Assignment due next class

- Read section 5.2 in the textbook.