1 Every sequentially compact metric space is separable

(a) Fix a natural number n.

Pick an arbitrary point x_1, then (if possible) a point x_2 outside $B_{1/n}(x_1)$, then (if possible) a point x_3 outside $B_{1/n}(x_1) \cup B_{1/n}(x_2)$, and so on.

Why does sequential compactness force this construction to stop after finitely many steps?

(b) The collection of all the points in part (a) for all n is a countable set.

(c) This countable set is dense.

2 Every sequentially compact metric space is second countable

(a) Assume that the space is separable.

(Another group is showing that every sequentially compact metric space is separable.)

(b) Take a countable dense set, and for each point x of that set, collect all the balls $B_{1/n}(x)$ as n runs over the natural numbers.

(c) Why is this collection of balls a countable collection?

(d) Why is this collection of balls a basis for the metric topology?

3 Every sequentially compact metric space is compact

(a) Another group is showing that sequentially compact metric spaces are second countable. So it suffices to show that every countable open cover, say $\{B_n\}_{n=1}^{\infty}$, has a finite subcover.

(b) If there is no finite subcover, then it is possible for each n to choose a point x_n outside $B_1 \cup \cdots \cup B_n$.

(c) Sequential compactness implies that this sequence of points has a subsequence converging to some limit, say x.

(d) There is some natural number j for which $x \in B_j$.

(e) Since B_j is open, some tail of the convergent subsequence lies inside B_j.

(f) By construction, $x_k \notin B_j$ when $k \geq j$. Contradiction.

4 Compactness and the finite-intersection property

Here is the statement of problem 7.2.3:

> Let (X, τ) be a compact space. If $\{F_i : i \in I\}$ is a family of closed subsets of X such that $\bigcap_{i\in I} F_i = \varnothing$, prove that there is a finite subfamily
>
> $$F_{i_1}, F_{i_2}, \ldots, F_{i_m} \quad \text{such that} \quad F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_m} = \varnothing.$$

(a) The implication about closed sets in this statement is the contrapositive of a property stated (but not proved) in class to be equivalent to compactness.

(b) Apply De Morgan's law for set complements to show that the indicated property of closed sets is equivalent to compactness.

5 Automatic continuity of the inverse function

Here is the statement of problem 7.2.6, an important theorem:

> Let $f : (X, \tau) \to (Y, \tau_1)$ be a continuous bijection. If (X, τ) is compact and (Y, τ_1) is Hausdorff, prove that f is a homeomorphism.

(a) Proposition 7.2.1 implies that the space Y is compact.

(b) The space X is necessarily Hausdorff: to separate points x_1 and x_2, find open sets that separate $f(x_1)$ and $f(x_2)$ and pull these sets back by f^{-1}.

(c) So both X and Y are compact Hausdorff spaces.

Propositions 7.2.4 and 7.2.5 imply that a subset of a compact Hausdorff space is compact if and only if closed.

(d) To solve the problem, what needs to be shown is that f^{-1} is continuous.

(e) Equivalently, what needs to be shown is that f is a closed mapping: namely, for every closed subset A of X, the image $f(A)$ is a closed subset of Y.

(f) In view of part (c), what needs to be shown is that f maps compact sets to compact sets. And this conclusion follows from Proposition 7.2.1.

6 Every compact Hausdorff space is normal

This statement is problem 7.2.10. The definition of "normal" (or T_4) can be found in problem 6.1.9. The meaning is that if A and B are two arbitrary disjoint closed sets, then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(a) Suppose $a \in A$ and $b \in B$. The Hausdorff condition produces disjoint open neighborhoods N_a and N_b such that $a \in N_a$ and $b \in N_b$.

(b) Keeping a fixed, let b vary over the set B. The neighborhoods N_b form an open cover of B.

(c) The set B is compact (Proposition 7.2.4), so there is a finite subcover.

(d) Take the corresponding finite number of neighborhoods of a and intersect them to get an open neighborhood of a that is disjoint from an open set containing B.

(e) Now let the point a vary. A finite number of the constructed neighborhoods cover the compact set A.

(f) The union of the sets constructed in part (e) can be taken to be U. What is the required open set V?