Final Examination

1. Suppose $X = \{1,2,3\}$, and $\mathcal{B} = \{\{1,2\},\{2,3\}\}$. Is the set $\mathcal{B}$
   (a) a topology on X?
   (b) a basis for a topology on X?
   (c) a subbasis for a topology on X?

   Explain why or why not.

   Solution. A topology must contain both $\varnothing$ and X, so $\mathcal{B}$ is not a topology.

   The intersection of two members of a basis must be expressible as a union of members of the basis. But $\{1,2\} \cap \{2,3\} = \{2\}$, and $\{2\}$ is not a union of members of $\mathcal{B}$. Thus $\mathcal{B}$ is not a basis for any topology.

   Every non-empty collection of non-empty sets is a subbasis for some topology on the union of the sets: namely, the intersections of finitely many members of the collection form a basis for some topology. In the case at hand, the collection $\mathcal{B}$ is a subbasis for the topology $\{\varnothing, \{2\}, \{1,2\}, \{2,3\}, X\}$.

2. Suppose X is the set $\{x, y, z\}$ equipped with the topology $\{\varnothing, \{x\}, \{x, y\}, X\}$. Determine
   (a) the interior of the singleton subset $\{y\}$, and
   (b) the closure of the singleton subset $\{y\}$.

   Solution. The interior of the set $\{y\}$ is the largest open subset of $\{y\}$: namely, $\varnothing$.

   The closure of the set $\{y\}$ is the smallest closed set containing $\{y\}$. The closed sets (the complements of open sets) are X, $\{y, z\}$, $\{z\}$, and $\varnothing$. The smallest closed set containing the set $\{y\}$ is therefore the set $\{y, z\}$.

3. Suppose $d(x, y) = \left|\log\left(\frac{x}{y}\right)\right|$. Is this function d a metric on the set of positive rational numbers? Explain why or why not. [Recall that $\log(x/y) = \log(x) - \log(y)$.]

   Solution. To see that d is a metric, simply check all the required properties.

   Evidently $d(x, y) = 0$ if and only if $\log(x/y) = 0$, which happens if and only if $x = y$. Being represented by an absolute value, the function d is never negative. And d is symmetric, since
   $$\left|\log\left(\frac{y}{x}\right)\right| = \left|-\log\left(\frac{x}{y}\right)\right| = \left|\log\left(\frac{x}{y}\right)\right|.$$
   Applying the triangle inequality for the Euclidean metric shows that
   $$\begin{aligned} d(x,z) = \left|\log\left(\frac{x}{z}\right)\right| &= |\log(x) - \log(y) + \log(y) - \log(z)| \\ &\le |\log(x) - \log(y)| + |\log(y) - \log(z)| = d(x, y) + d(y, z), \end{aligned}$$
   so the triangle inequality holds for d.

Final Examination

4. The topological space in this problem is $\mathbb{R}$, the set of real numbers, equipped with the standard Euclidean topology.

   (a) Give an example of a subset of $\mathbb{R}$ that is compact but not connected.

   (b) Give an example of a subset of $\mathbb{R}$ that is connected but not compact.

   Solution. One example of a subset of $\mathbb{R}$ that is compact but not connected is the two-point set $\{-1, 1\}$. This set is disconnected, being covered by the two disjoint open sets $(-\infty, 0)$ and $(0, \infty)$. And every finite set of points is compact.

   One example of a subset of $\mathbb{R}$ that is connected but not compact is the interval $(0, 1)$. A standard theorem says that intervals are connected (even path-connected) with respect to the Euclidean topology. And the Heine–Borel theorem implies that the interval $(0, 1)$ is not compact (since not closed). Another example of a subset of $\mathbb{R}$ that is connected but not compact is the whole space $\mathbb{R}$ itself.

5. Suppose $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is the function defined as follows: $f(x) = \begin{cases} 1/x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

   (a) Give an example of a topology τ with respect to which the function f is continuous.

   (b) Give an example of a topology τ with respect to which f is not continuous.

   Solution. Here are three examples of topologies that make f continuous.

   (1) If $\mathbb{R}$ is given the discrete topology, then f is continuous. Indeed, every function is continuous when the domain of the function has the discrete topology. (The inverse image of every open set is open, for every subset of the domain is open.)

   (2) If $\mathbb{R}$ is given the indiscrete topology, then f is continuous. Indeed, every function is continuous when the codomain has the indiscrete topology. (The inverse image of the whole space is the whole space, and the inverse image of the empty set is the empty set; both of these inverse images are always open in the domain.)

   (3) If $\mathbb{R}$ is given the finite-closed (cofinite) topology, then f is continuous. Indeed, the specified function f is bijective, so the inverse image of every finite set is finite. In other words, the inverse image of every closed set is closed. Equivalently, the inverse image of every open set is open.

   Here are three examples of topologies that make f discontinuous.

   (1) If $\mathbb{R}$ is given the Euclidean topology, then f is not continuous. Indeed, $f^{-1}((-1, 1)) = \{0\} \cup (1, \infty) \cup (-\infty, -1)$, and this inverse image is not open in $\mathbb{R}$ (since the set contains no neighborhood of 0). Since there is an open set whose inverse image is not open, the function f is not continuous.

   (2) By the same reasoning, the function f is discontinuous with respect the Sorgenfrey topology.

(3) If $\tau = \{\varnothing, \{5\}, \mathbb{R}\}$, then f is discontinuous. Indeed, the set $\{5\}$ is open in the codomain, but $f^{-1}(\{5\}) = \{1/5\}$, and the set $\{1/5\}$ is not open in the domain.

6. True/false: For each of the following statements, say whether the sentence is true or false (exclusive "or"). If the statement is false, give a counterexample; if the statement is true, give a brief explanation why.
   (a) Every path-connected topological space is connected.
   (b) Every metric space is a Hausdorff space (that is, T_2 space) with respect to the topology induced by the metric.
   (c) A subset E of a topological space X is dense in X if and only if the following property holds: $E \cap U \neq \varnothing$ for every nonempty open set U.

Solution. All three statements are true.

A standard theorem (Proposition 5.2.6) says that a path-connected space is connected. The idea is that a path is a connected set (being the continuous image of the connected set $[0, 1]$), so if every two points can be joined by a path, then every two points lie in the same component of the space. [But the converse statement is false: not every connected space is path-connected. The topologist's sine curve is the standard counterexample.]

Another standard theorem (Proposition 6.1.25) says that metric spaces are Hausdorff spaces. Indeed, if x and y are distinct points of the space, then open balls centered at these points with radius $d(x, y)/2$ are disjoint neighborhoods of x and y.

The standard definition of density (Definition 3.1.13) is that the closure of E is the whole space X. The stated property of E is equivalent (Proposition 3.1.15). Indeed, if there were a non-empty open set U such that $E \cap U = \varnothing$, then $X \setminus U$ would be a closed set containing E, so the closure of E would be a subset of $X \setminus U$, hence a proper subset of X. Conversely, if E were not dense, then there would be some point x outside the closure of E, hence there would be a neighborhood U of x that does not intersect E.

Optional Extra Credit Problem

Determine the homeomorphism classes of intervals in $\mathbb{R}$ with respect to the Sorgenfrey topology. In other words, if intervals are equipped with the subspace topology induced by the Sorgenfrey topology on $\mathbb{R}$, then which intervals are homeomorphic to each other?

Solution. First dispense with some special cases. The definition of "interval" in the textbook seems to admit the empty set as a degenerate interval. The empty set evidently is the single element of its homeomorphism class. The textbook explicitly includes singletons as intervals. All singletons are homeomorphic to each other, and no singleton set can be homeomorphic to a non-singleton set (since a singleton is not even bijectively equivalent to a non-singleton). Accordingly, what needs to be discussed is the case of intervals containing at least two points (hence infinitely many points).

The claim is that all such genuine intervals belong to precisely two homeomorphism classes. One class consists of the intervals that contain a maximal point, and the other class consists of the intervals that do not contain a maximal point. The intervals in the first homeomorphism class have the forms $[a, b]$ (where $a < b$) and $(a, b]$ and $(-\infty, b]$. The intervals in the second homeomorphism class have the forms (a, b), $[a, b)$, $(-\infty, b)$, (a, ∞), $[a, \infty)$, and $(-\infty, \infty)$.

The proof is based on two observations. First, both translation (addition of a real number) and dilation (multiplication by a positive real number) are homeomorphisms of the Sorgenfrey line, for these operations map every Sorgenfrey basic open set $[a, b)$ bijectively to another such set. Thus each interval can be mapped homeomorphically to a standardized interval with prescribed endpoints. In other words, the claim amounts to showing that (i) the intervals $[0, 1]$ and $(0, 1]$ and $(-\infty, 0]$ are Sorgenfrey-homeomorphic to each other, (ii) the intervals $(0, 1)$, $[0, 1)$, $(-\infty, 0)$, $(0, \infty)$, $[0, \infty)$, and $(-\infty, \infty)$ are Sorgenfrey-homeomorphic to each other, and (iii) the intervals $[0, 1]$ and $(0, 1)$ are not Sorgenfrey-homeomorphic to each other.

The second observation is that if X is a disconnected topological space that is expressed as the union of two non-empty, disjoint, open sets A and B, and if f is a function from X to some other topological space, and if the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous. Indeed, continuity of f means that $f^{-1}(U)$ is open for every open subset U of the codomain, and

$$f^{-1}(U) = \left(f^{-1}(U) \cap A\right) \bigcup \left(f^{-1}(U) \cap B\right) = (f|_A)^{-1}(U) \bigcup (f|_B)^{-1}(U).$$

The right-hand side is an open subset of X due to the assumption of continuity of the restrictions $f|_A$ and $f|_B$ and the openness of the sets A and B. The same principle holds (for the same reason) if X is expressed as the union of more than two (even infinitely many) pairwise disjoint open subsets.

Now each of the intervals $(0, 1)$ and $[0, 1)$ can be expressed as the union of countably many pairwise disjoint Sorgenfrey basic open sets: namely,

$$(0,1) = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right), \qquad \text{and} \qquad [0,1) = \bigcup_{n=1}^{\infty} \left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right).$$

The intervals $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$ and $\left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right)$ have the same length, so the translation that sends x to $x - \frac{1}{2^n} + 1 - \frac{1}{2^{n-1}}$ is a homeomorphism between them. By the second observation above, a homeomorphism between $(0, 1)$ and $[0, 1)$ arises by pasting together these homeomorphisms between the countably many subintervals composing $(0, 1)$ and $[0, 1)$.

All of the other intervals in the proposed homeomorphism class of $(0, 1)$ also can be expressed as countable unions of pairwise disjoint Sorgenfrey basic open sets: namely,

$$(-\infty, 0) = \bigcup_{n=1}^{\infty} [-n, -n+1), \qquad (0, \infty) = \left\{ \bigcup_{n=1}^{\infty} \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right) \right\} \bigcup \left\{ \bigcup_{k=1}^{\infty} [k, k+1) \right\},$$

$$[0, \infty) = \bigcup_{n=1}^{\infty} [n-1, n), \qquad (-\infty, \infty) = \bigcup_{n=-\infty}^{\infty} [n, n+1).$$

The same reasoning showing that $(0, 1)$ and $[0, 1)$ are Sorgenfrey-homeomorphic to each other therefore shows that all of the indicated intervals are Sorgenfrey-homeomorphic to $(0, 1)$. This conclusion establishes part (ii) of the claim.

In the Sorgenfrey-subspace topology, the singleton $\{1\}$ is an open subset of $[0, 1]$, being equal to the intersection of $[0, 1]$ with the Sorgenfrey-open set $[1, 2)$. Similarly, singleton $\{1\}$ is an open subset of $(0, 1]$ in the Sorgenfrey-subspace topology. The intervals $[0, 1)$ and $(0, 1)$ are already known to be Sorgenfrey-homeomorphic, so the second observation above implies that the homeomorphism can be extended to a homeomorphism between $[0, 1]$ and $(0, 1]$ simply by making 1 correspond to 1. By the same reasoning, since $[0, 1)$ and $(-\infty, 0)$ are already known to be Sorgenfrey-homeomorphic, extending the homeomorphism by sending 1 to 0 produces a Sorgenfrey-homeomorphism between the intervals $[0, 1]$ and $(-\infty, 0]$. So part (i) of the claim holds.

Finally, why are the intervals $[0, 1]$ and $(0, 1)$ not Sorgenfrey-homeomorphic to each other? Although the singleton $\{1\}$ is an open subset of $[0, 1]$ in the Sorgenfrey-subspace topology, no singleton is an open subset of $(0, 1)$. Indeed, the intersection of a Sorgenfrey basic open set $[a, b)$ with $(0, 1)$ is either empty or a nontrivial interval containing infinitely many points. Since homeomorphisms preserve the structure of open sets, there cannot be a Sorgenfrey-homeomorphism between $[0, 1]$ and $(0, 1)$, for there is no open singleton available to be the image of $\{1\}$. This conclusion establishes part (iii) of the claim and thereby finishes the solution.

Remark. The corresponding problem for the Euclidean topology is discussed in the textbook in Corollary 4.3.7 and Exercise 4.3.1.