Normal families of holomorphic functions

The goal of this exercise is to demonstrate that you understand and can apply the statement of Montel's theorem and the definitions of normal family and locally bounded family.

A family $\mathcal{F}$ of continuous functions on an open set U in $\mathbb{C}$ is called a *normal family* if every sequence $\{f_n\}$ of functions in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of U. A family $\mathcal{F}$ is called *locally bounded* or *bounded on compact sets* if for every compact set K contained in U there is a constant M (depending on K) such that $|f(z)| \le M$ for every z in K and every function f in the family $\mathcal{F}$.

The Arzelà-Ascoli theorem says that a family $\mathcal{F}$ of continuous functions on an open set U in $\mathbb{C}$ is a normal family if and only if the family $\mathcal{F}$ is equicontinuous (for every z_0 in U and for every positive ϵ there is a positive δ such that $|f(z) - f(z_0)| < \epsilon$ for every function f in $\mathcal{F}$ when $|z - z_0| < \delta$) and pointwise bounded (for every z_0 in U the set $\{\, f(z_0) : f \in \mathcal{F} \,\}$ is a bounded subset of $\mathbb{C}$). *Montel's theorem* says that a family $\mathcal{F}$ of *holomorphic* functions is normal if and only if it is locally bounded.

Suppose that $\mathcal{F}$ is a normal family of holomorphic functions on an open set U in $\mathbb{C}$. By operating on each function of the family, you can produce a new family. (For example, a homework exercise showed that differentiating a family preserves normality.) For which of the following operations is the new family of holomorphic functions again a normal family?

Paul Montel
1876–1975

1. Multiplication by the variable z.
2. $f \mapsto 1/f$.
3. Anti-differentiation.
4. An operation of your choosing.

The statement of the exercise is deliberately somewhat vague. As a part of your solution, indicate what additional assumptions or restrictions you think are reasonable to impose.